Computational Geometry
csci3250

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Motion Planning

Input:

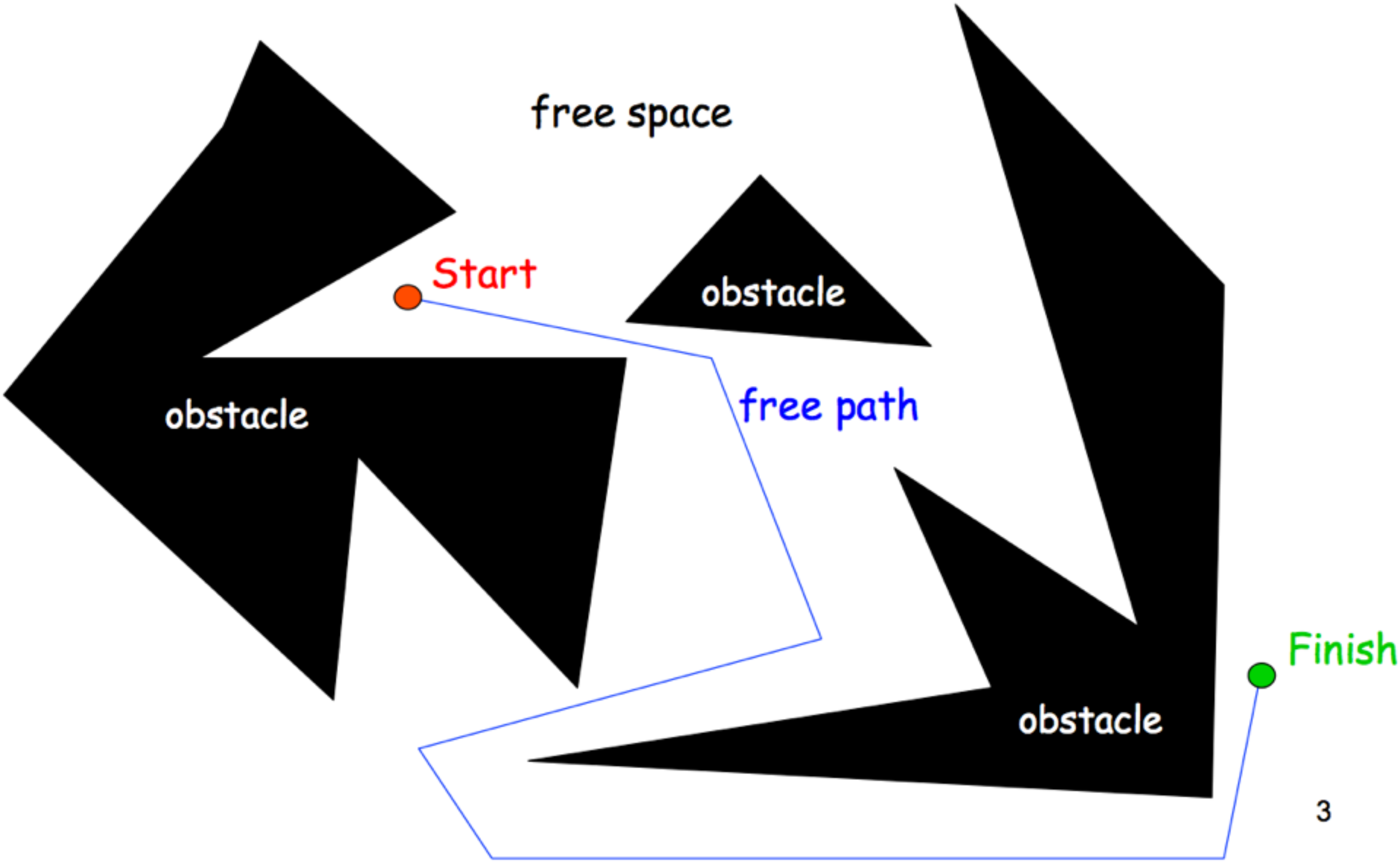
- a robot R and a set of obstacles $S = \{O_1, O_2, \ldots\}$
- start position p_{start}
- end position p_{end}

Find a path from start to end (that optimizes some objective function).

Parameters:

- geometry of obstacles (polygons, disks, convex, non-convex, etc)
- geometry of robot (point, polygon, disc)
- robot movement (dof)
- objective function to minimize (euclidian distance, nb turns, etc)
- 2d, 3d
- static vs dynamic environment
- exact vs approximate algorithm
- known vs unknown map

Problem



screenshot from: ai.stanford.edu/~latombe/cs26n/2012/slides/**point-robot**-bug.ppt

Motion Planning

Ideally we want a planner to be complete and optimal.

- A planner is complete:
  - it always finds a path when a path exists
- A planner is optimal:
  - it finds an optimal path

A Preview of Approaches

Exact (geometric/combinatorial)

- Roadmaps
  - from trapezoidal decomposition
  - Visibility graphs

Approximate

- sampling-based
- approximate space decomposition

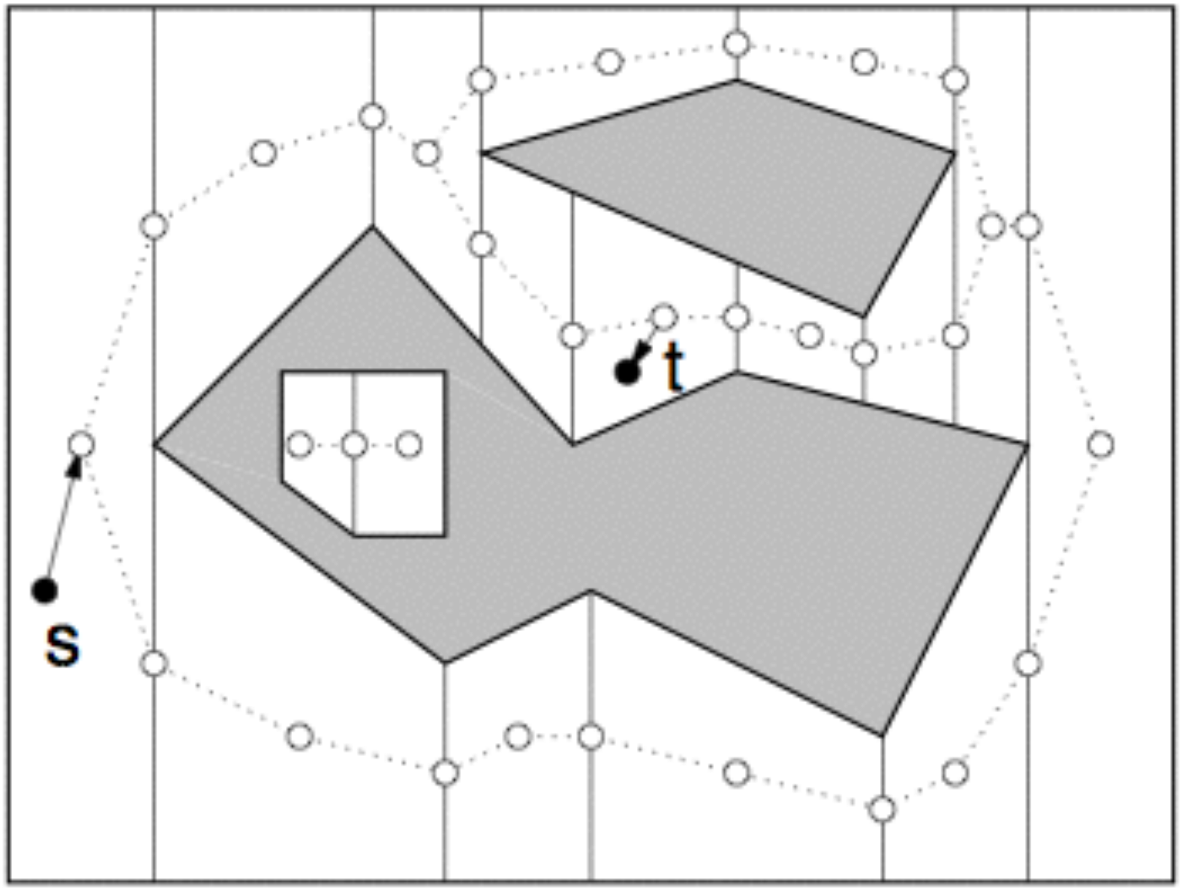
Today

Exact (geometric/combinatorial)

- Point robot in 2D
  - Paths: Roadmaps via trapezoid decomposition
  - Shortest paths: Visibility graph
- Polygon robot in 2D
  - Translation only
  - Handling rotations

Point robot in 2D

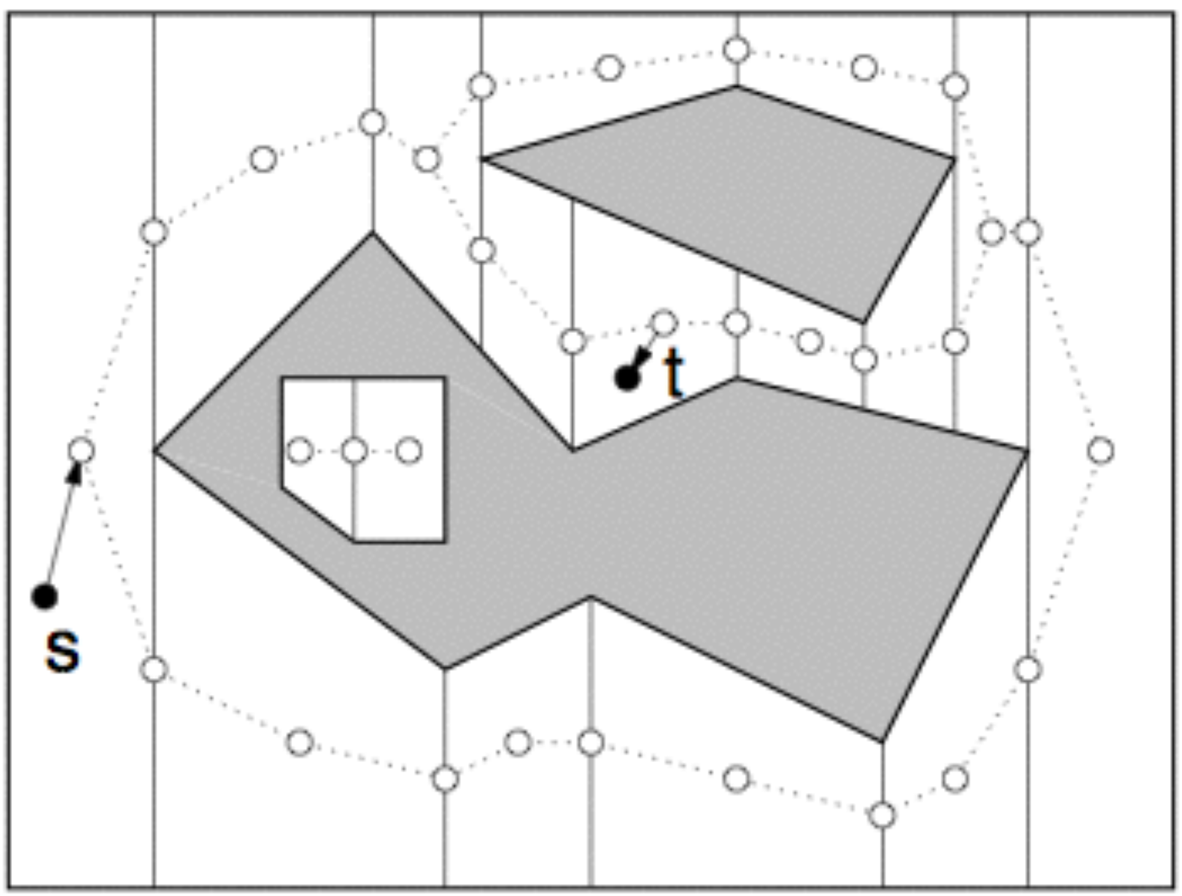
- General idea
  - Compute free space
  - Compute a representation of free space
  - Build a graph of free space
  - Search graph to find path <---------- Reduce motion planning to graph search



n = complexity of obstacles
(total number of edges)

Point robot in 2D

- General idea
  - Compute free space
  - Compute a representation of free space: trapezoidal decomposition, size O(n), O(n lg n) time
  - Build a graph of free space : size O(n), O(n) time
  - Search graph to find path: BFS in O(n) time

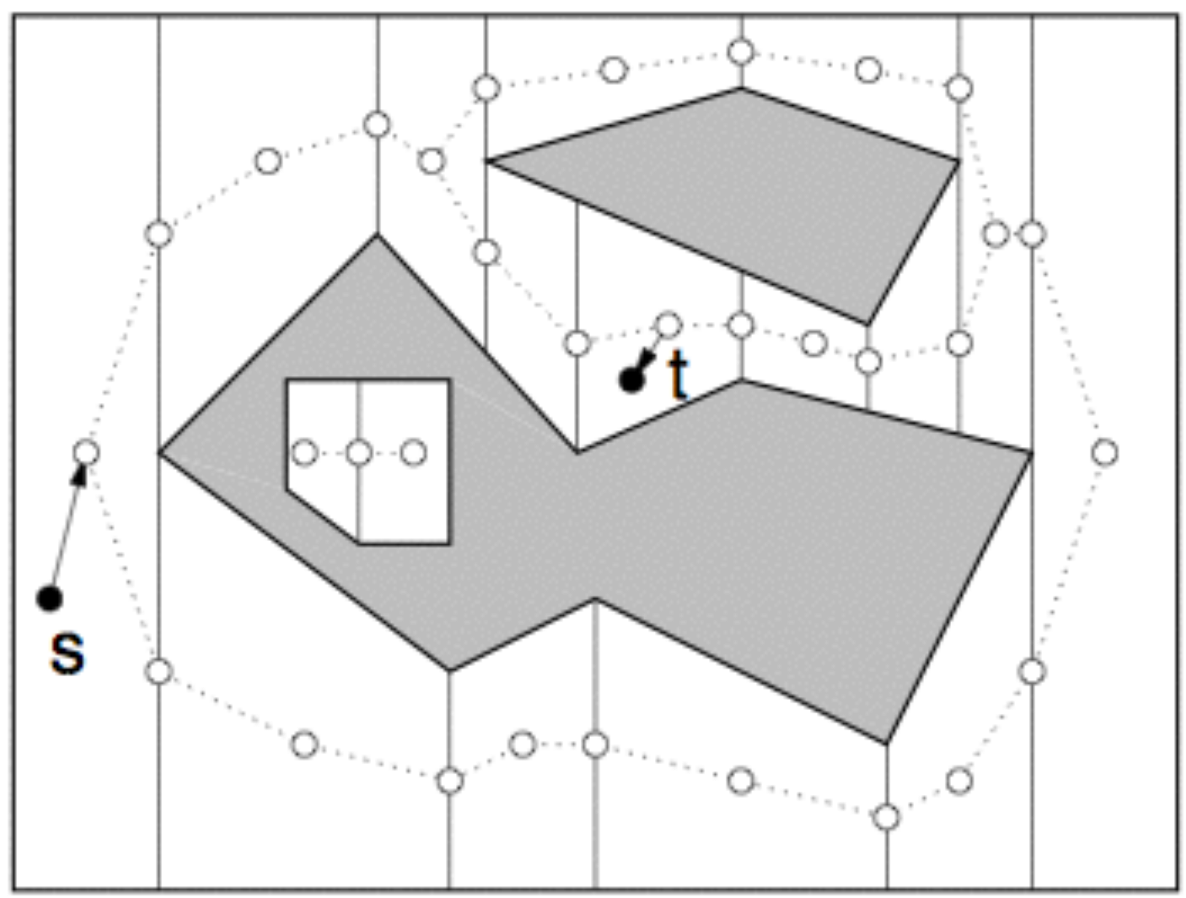


n = complexity of obstacles (total number of edges)

Point robot in 2D

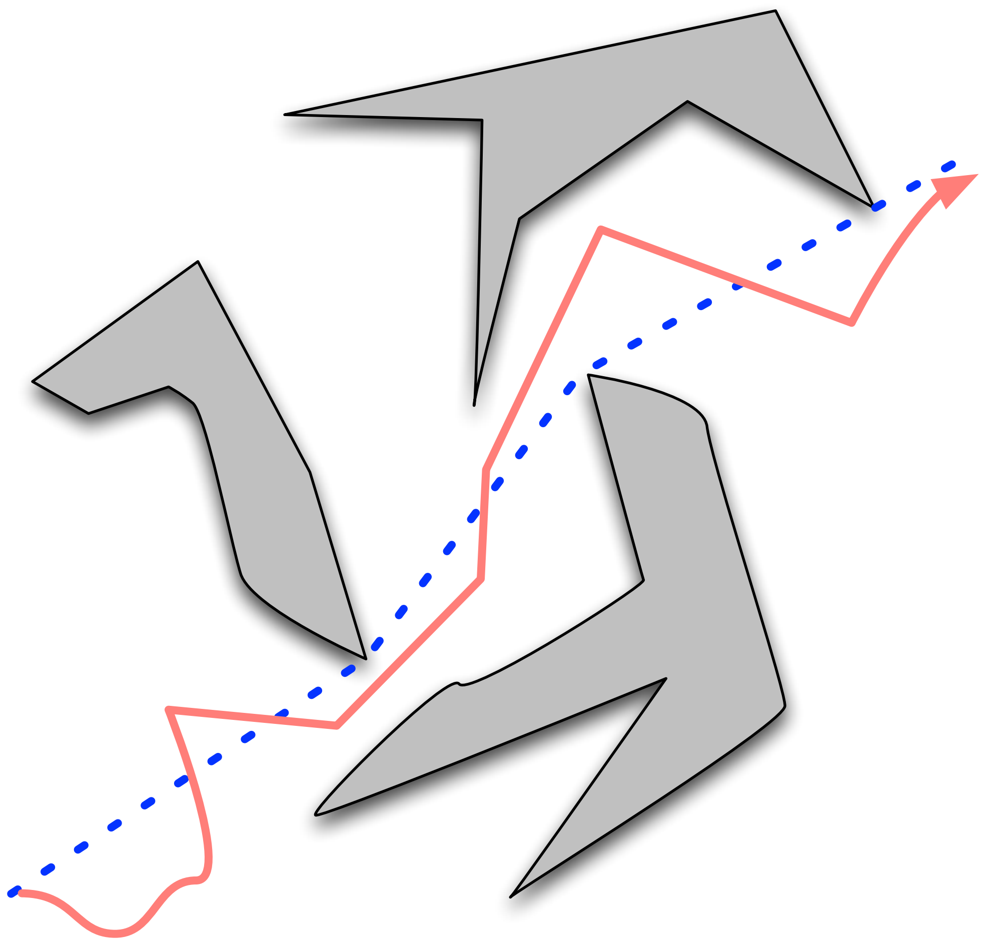
Result:

Let R be a point robot moving among a set of polygonal obstacles in 2D with n edges in total. We can pre-process S in O(n lg n) expected time such that, between any start and goal position, a collision-free path for R can be computed in O(n) time, if it exists.



Note: complete, but not optimal

Point robot in 2D

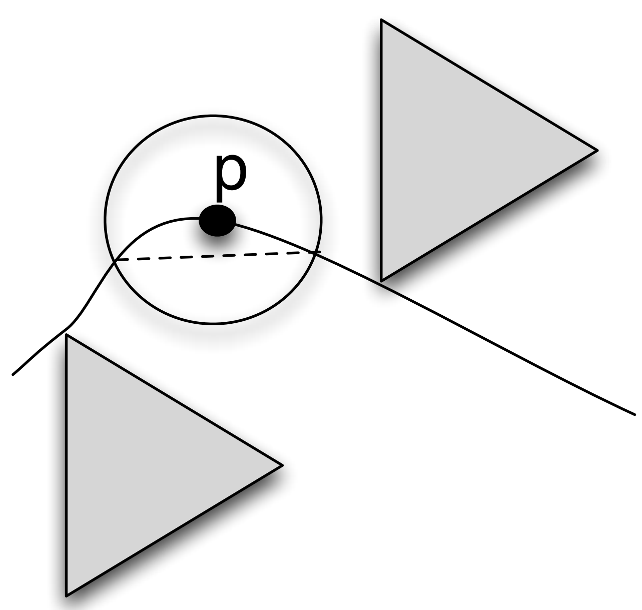


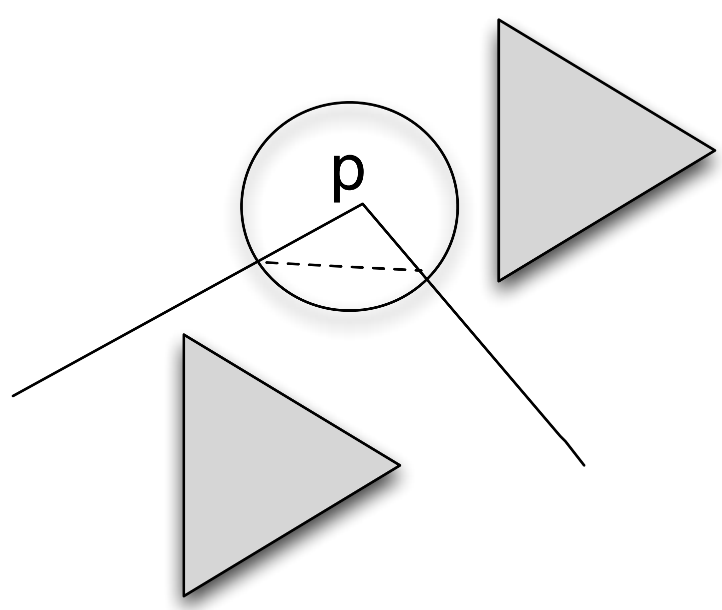
What if we wanted the shortest path p_{start} to p_{goal}?

Shortest paths for point robot in 2D

Any shortest path among a set S of disjoint polygonal obstacles:

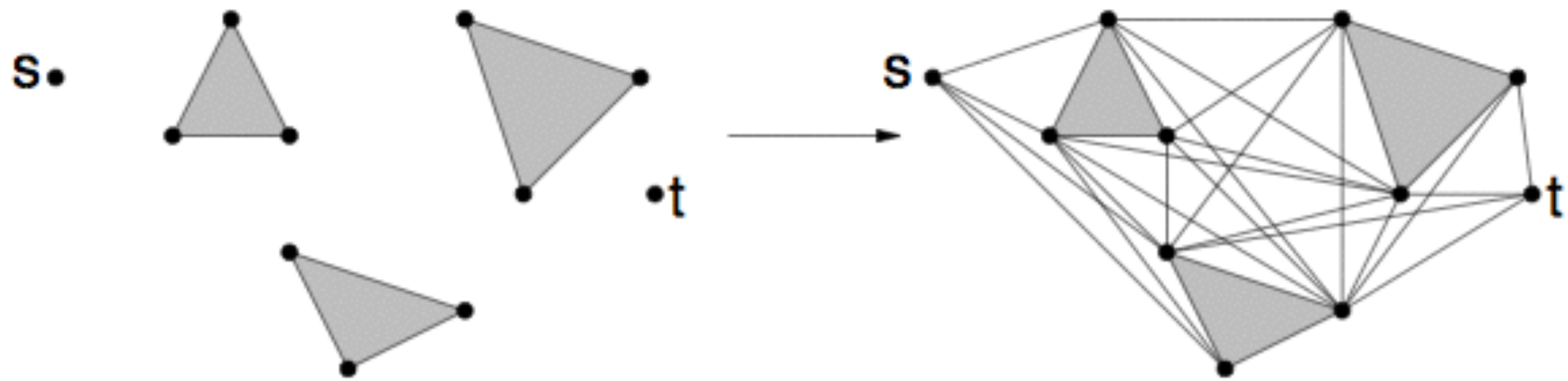
1. is a polygonal path (that is, not curved)
2. its vertices are the vertices of S.



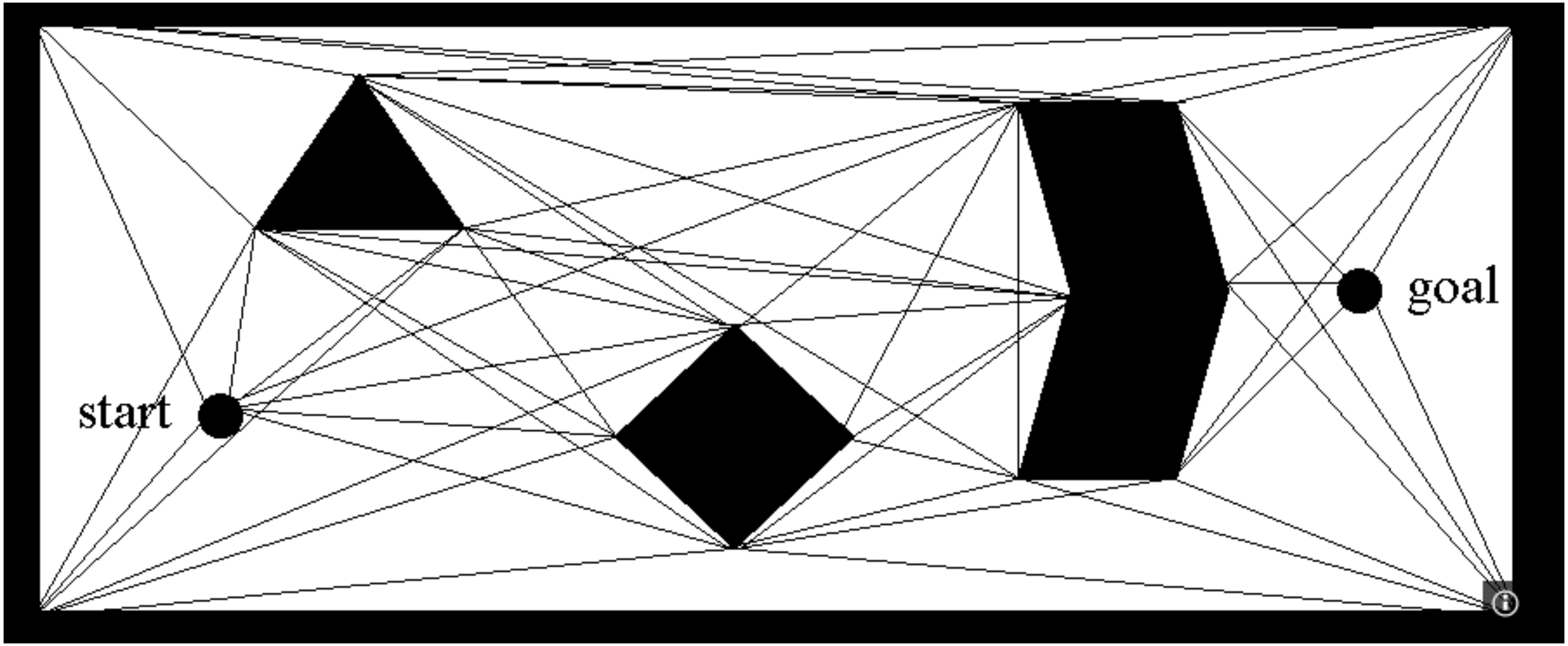


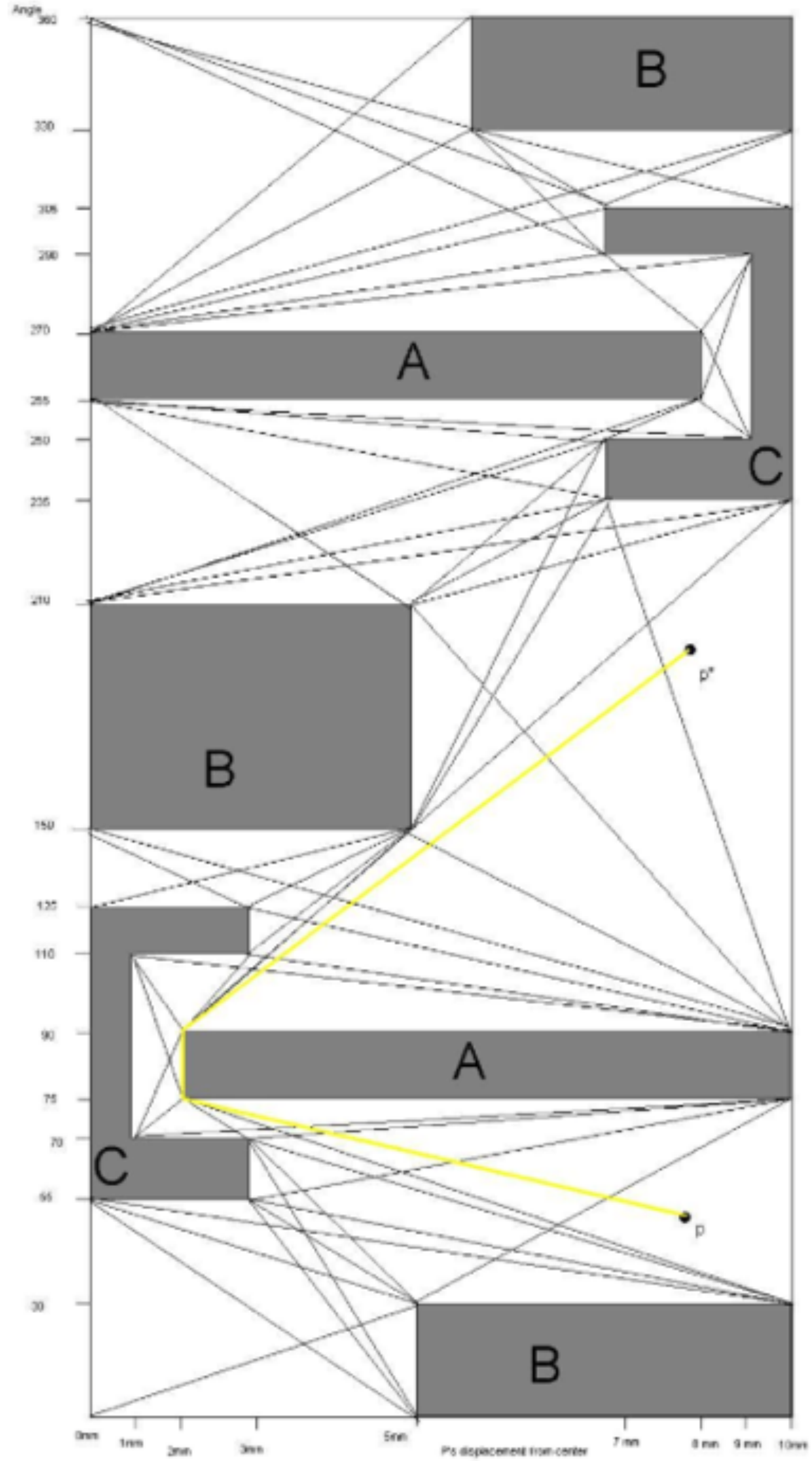
Shortest paths for point robot in 2D

- Idea: Build the visibility graph
  - all possible ways to travel between the vertices of the obstacles
- Claim: any shortest path must be a path in the VG



Shortest paths for point robot in 2D




Angle
360
330
305
290
270
255
250
235
210
150
135
110
90
75
70
55
30
B
A
C
B
C
A
B
p*
p
0mm
1mm
2mm
3mm
5mm
7 mm
8 mm
9 mm
10mm

n = complexity of obstacles
(total number of edges)

Shortest paths for point robot in 2D

Algorithm

- Compute visibility graph
    - V = {set of vertices of obstacles + p_{start} + p_{end}}
- SSSP (Dijkstra) in VG

- How big is VG and how long does it take to compute it?

n = complexity of obstacles
(total number of edges)

Shortest paths for point robot in 2D

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Shortest paths for point robot in 2D

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- Complexity of VG

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Shortest paths for point robot in 2D

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- Compute visibility graph
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- Complexity of VG
  - $V_{VG} = O(n)$, $E_{VG}=O(n^2)$ <-------- can have quadratic size

n = complexity of obstacles
(total number of edges)

Shortest paths for point robot in 2D

Algorithm

- Compute visibility graph
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- Computing VG

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- Computing VG
  - naive: for each edge, check if intersects any obstacle. $O(n^3)$

n = complexity of obstacles (total number of edges)

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  - improved: $O(n \lg n)$ per vertex, $O(n^2 \lg n)$ total

n = complexity of obstacles (total number of edges)

Shortest paths for point robot in 2D

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- Compute visibility graph
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    - $V_{VG} = O(n)$, $E_{VG}=O(n^2)$ <-------- can have quadratic size
- Computing VG
    - naive: for each edge, check if intersects any obstacle. $O(n^3)$
    - improved: $O(n \lg n)$ per vertex, $O(n^2 \lg n)$ total
- Dijkstra on VG: $O(E_{VG} \lg n) = O(n^2 \lg n)$

n = complexity of obstacles (total number of edges)

Shortest paths for point robot in 2D

Algorithm

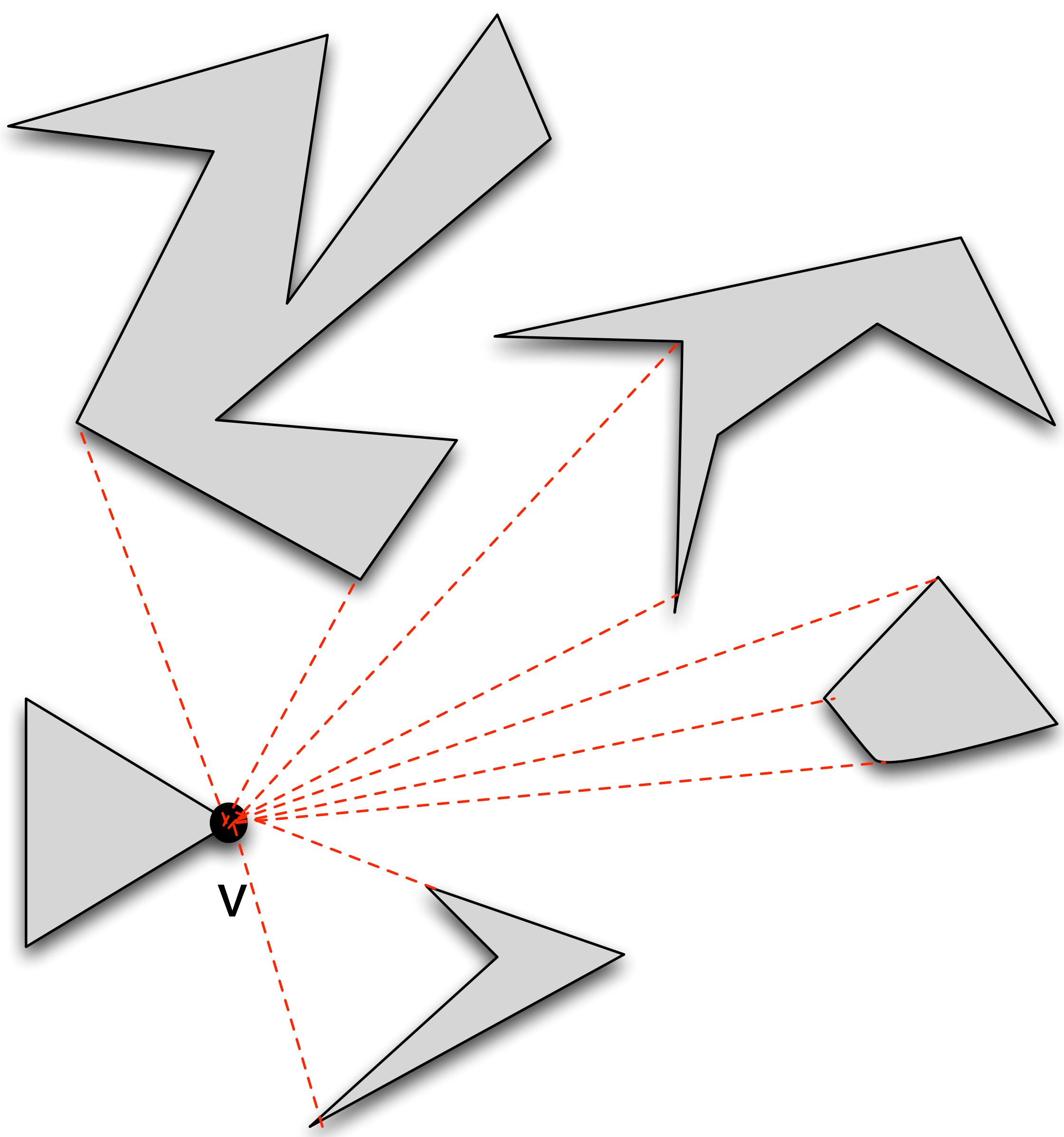
- Compute visibility graph <------- $O(n^2 \lg n)$
- SSSP (Dijkstra) in VG <------- $O(E_{VG} \lg n)$

Theorem:

A shortest path of two points among a set of polygonal obstacles with n edges in total can be computed in $O(E_{VG} \lg n) = O(n^2 \lg n)$ time.

Improved computation of VG

- Idea
    - for every vertex v: compute all vertices visible from v



Improved computation of VG

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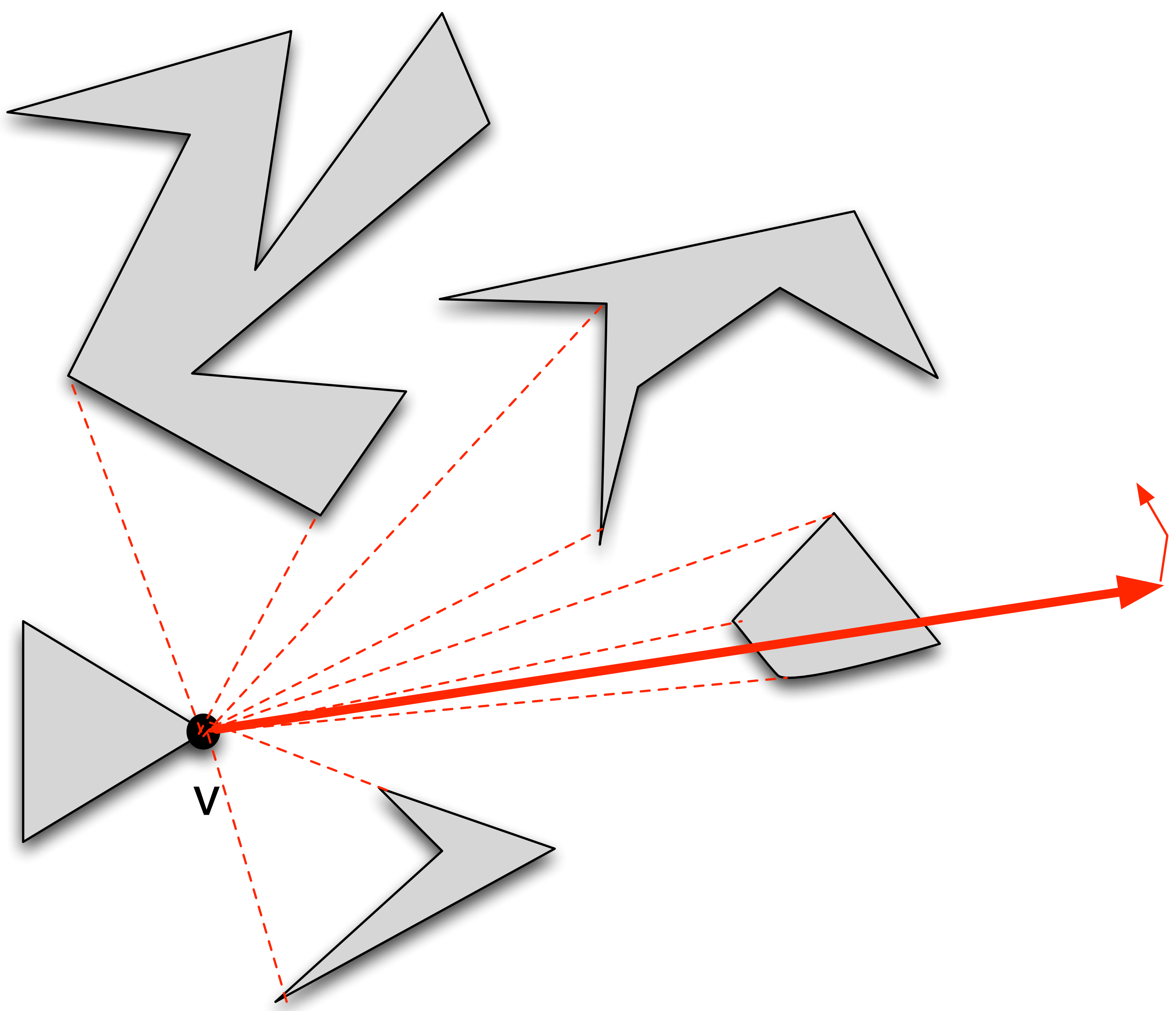


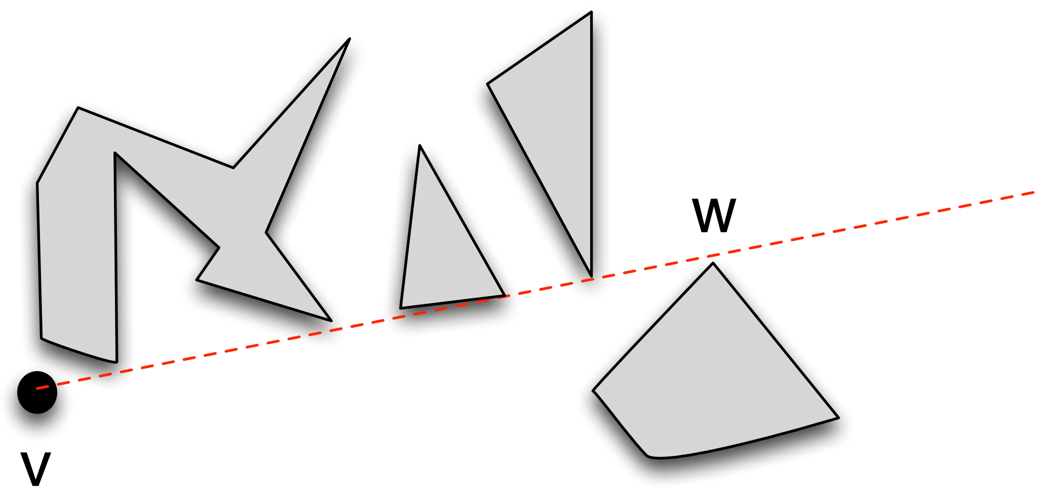
Improved computation of VG

- Idea
  - for every vertex v: compute all vertices visible from v

active structure stores all edges intersected by sweep line,
ordered by distance from v

Improved computation of VG

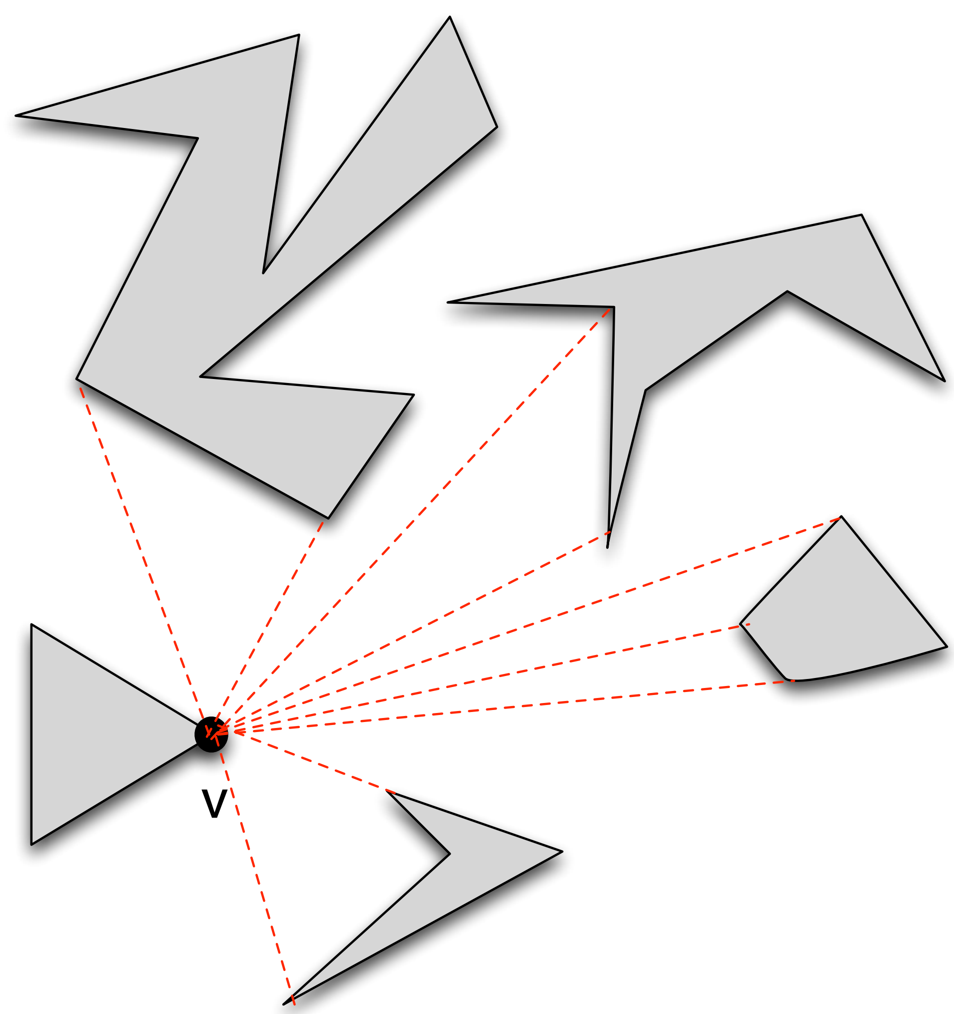
- Idea
  - for every vertex v: compute all vertices visible from v



w visible if vw does not intersect the interior of any obstacle

Improved computation of VG

- Idea
  - for every vertex v: compute all vertices visible from v
  - Overall: $n \times n \lg n = O(n^2 \lg n)$



Motion planning via visibility graph

Comments

- Is it optimal? Is it complete?
- VG needs to be computed only once, so we can think of it as pre-processing
- VG may be large ==> doomed to quadratic complexity

History/overview:

- Quadratic barrier broken by Joe Mitchell: SP of a point robot moving in 2D can be computed in $O(n^{5/3 + eps})$
- Hershberger and Suri [1993]: SP of a point robot moving in 2D can be computed in O(n lg n) ("continuous Dijkstra" approach)
- Special cases can be solved faster:
    - e.g. SP inside a simple polygon w/o holes: O(n) time

Shortest paths in 3D

- VG does not generalize to 3D
- SP in 3D much harder: no easy way to discretize the problem
  - inflection points of SP are not restricted to vertices of S, can be inside edges
  - 3D shortest paths among polyhedral obstacles is NP-complete
  - Complete planning in 3D is hopeless

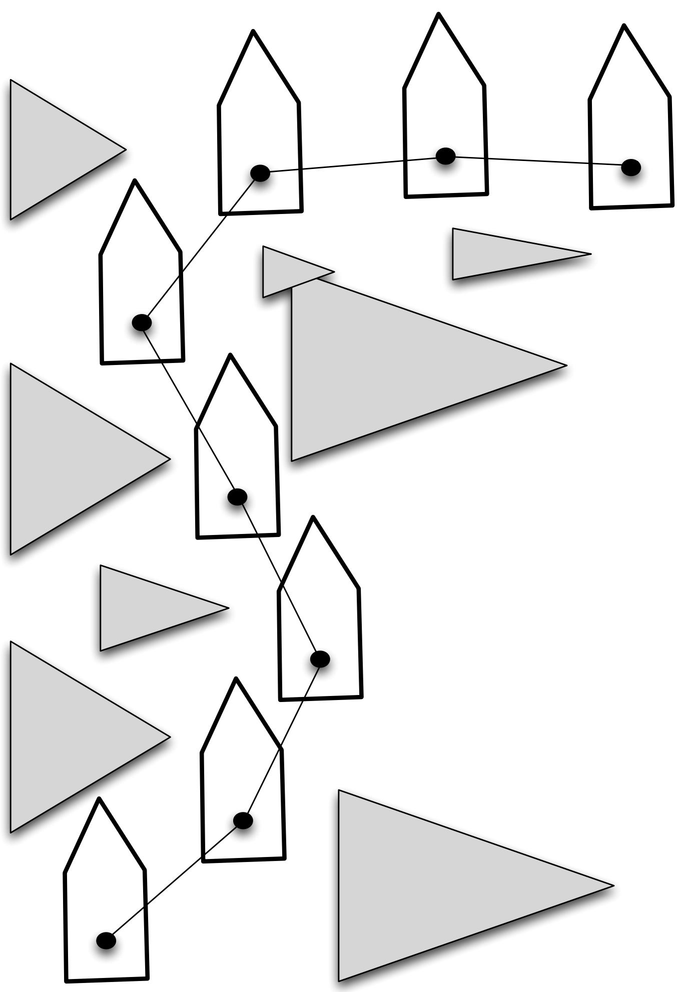
Today

Exact (geometric/combinatorial)

- Point robot in 2D
  - Paths: Roadmaps via trapezoid decomposition
  - Shortest paths: Visibility graph
- Polygon robot in 2D
  - Translation only
  - Handling rotations

Convex polygon moving in 2D

- How can the robot move?
  - Translation only
  - Translation + rotation



screenshot from internet

Work/physical space

- Space where robot moves around
- A placement of robot is specified by the degrees of freedom (dof) of the robot

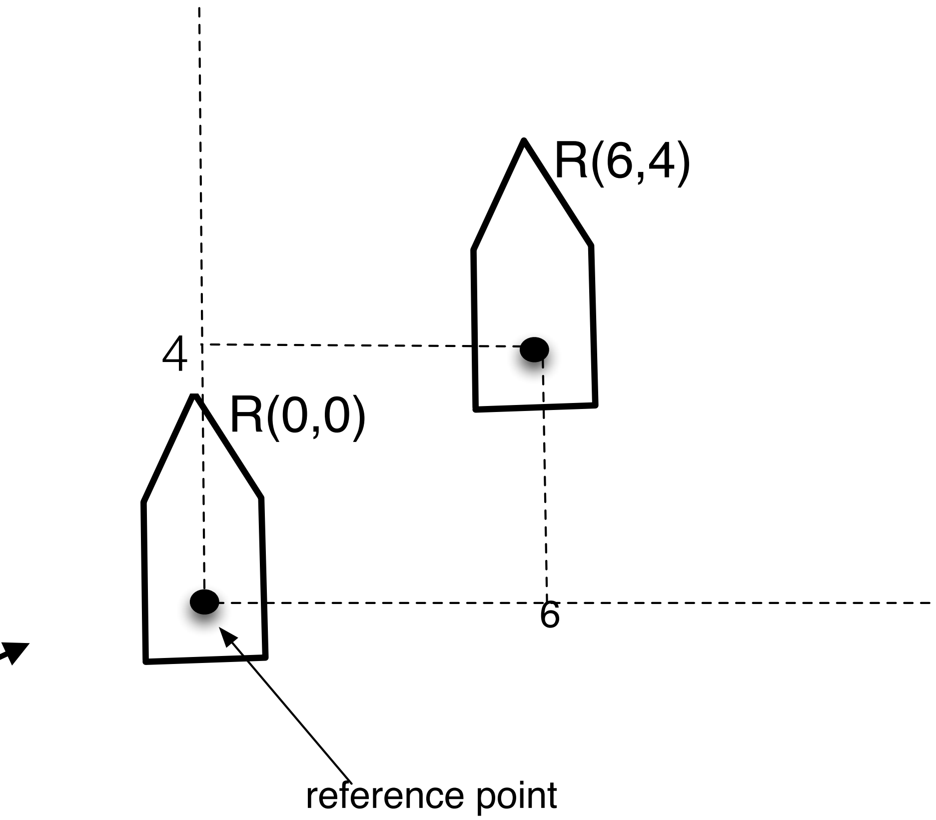
  - Examples:

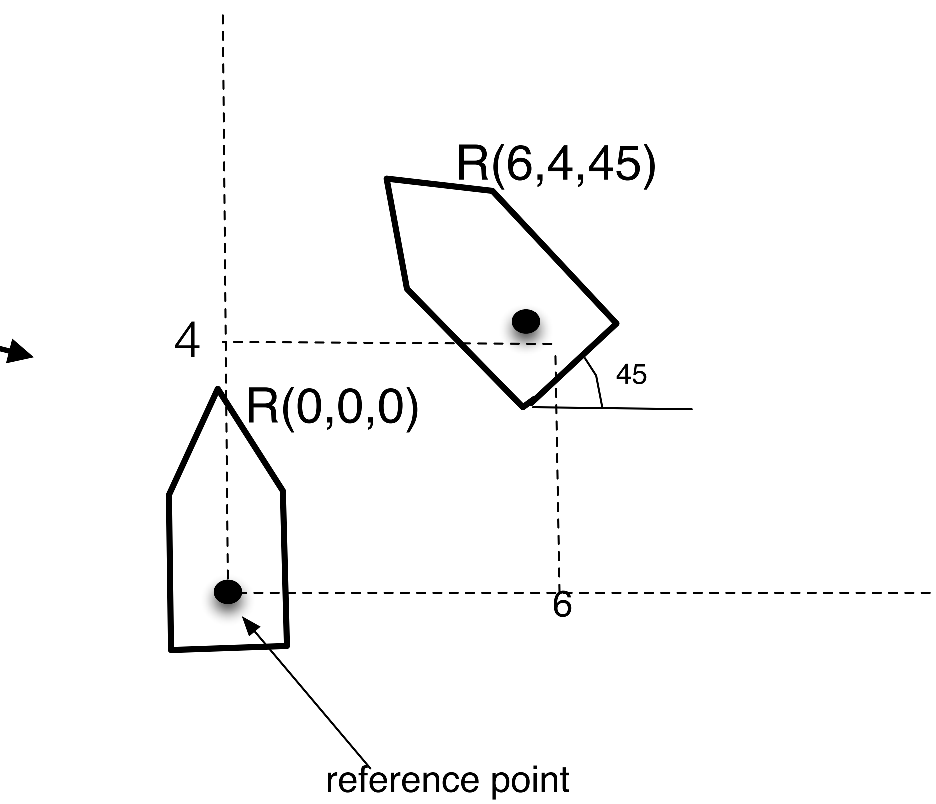
    2D: R(x,y)

    $R(x,y,\theta)$

    2D: $R(x,y,\theta_1,\theta_2)$

    3D: R(x,y,z)



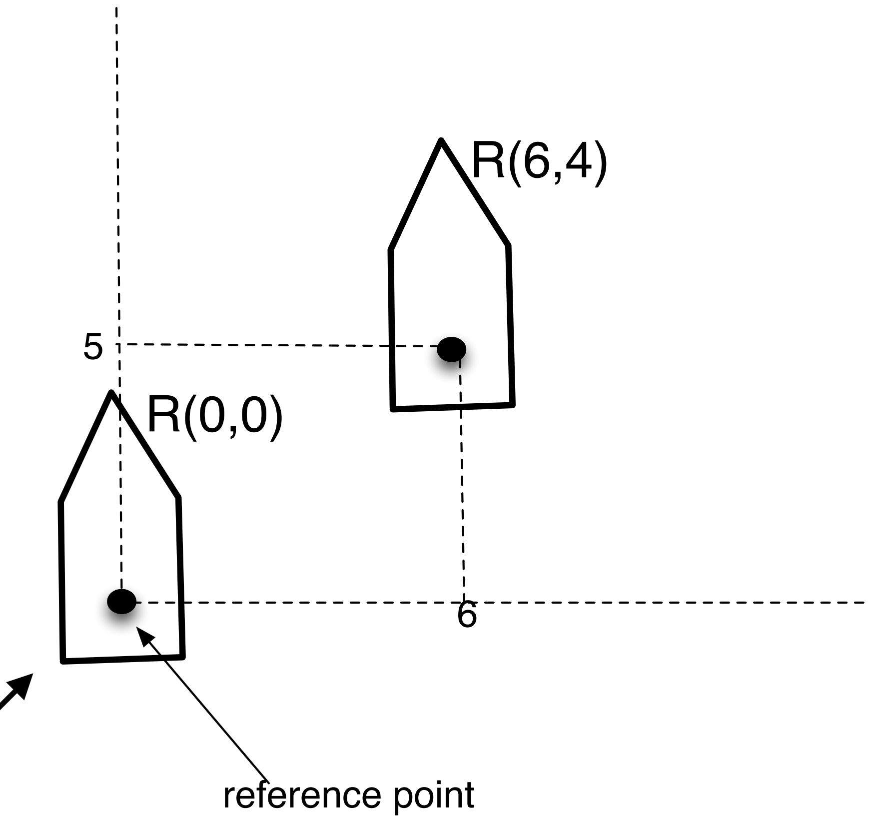


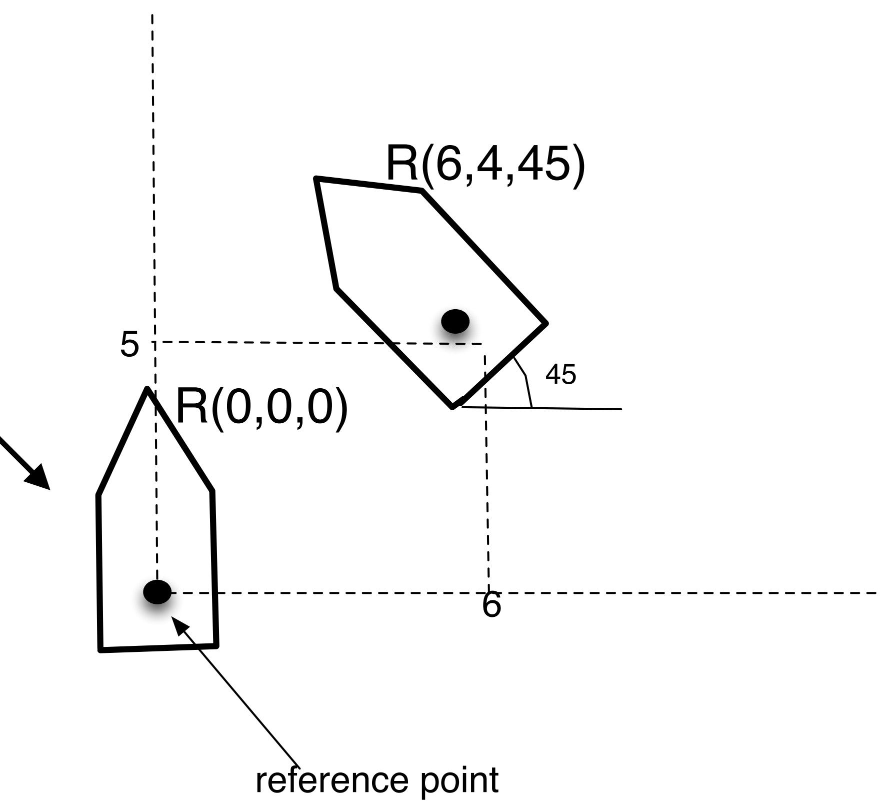
Configuration space (C-space)

- A point in C-space corresponds to placement of the robot in physical space
- The parametric space of the robot = space of all possible placements of the robot
  - Examples:

    2D, translation only <-> **R(x,y)**

    2D, transl.+ rot. <-> **R(x,y, theta)**





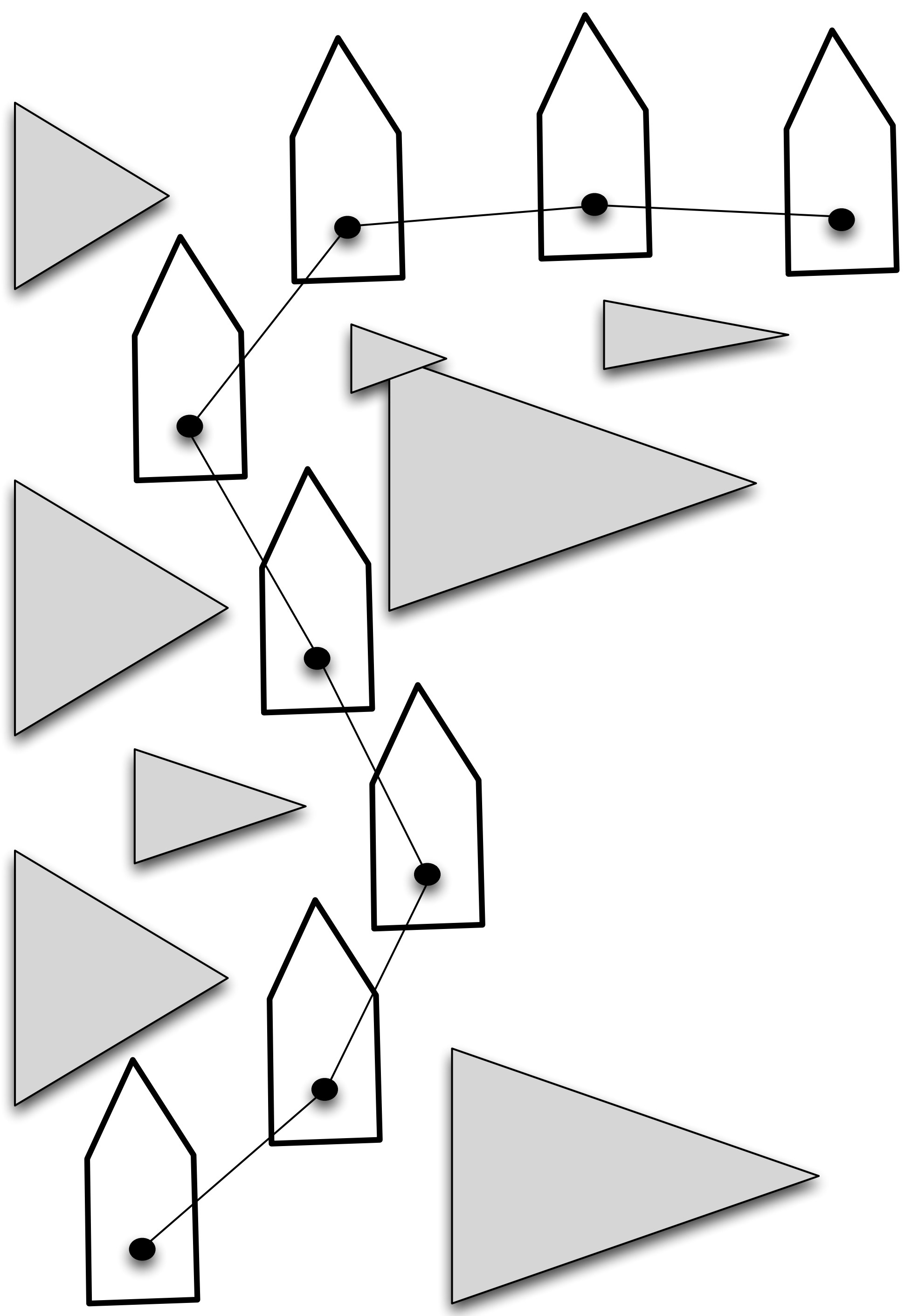
- A path for R corresponds to a path in C-space

Physical Space and C-space

robot	physical space	C-space
polygon, (translation only)	2D	2D
polygon, (translation + rotations)	2D	3D
polygon (translation, rotations)	3D	6D
Robot arm with joints	3D	#DOF

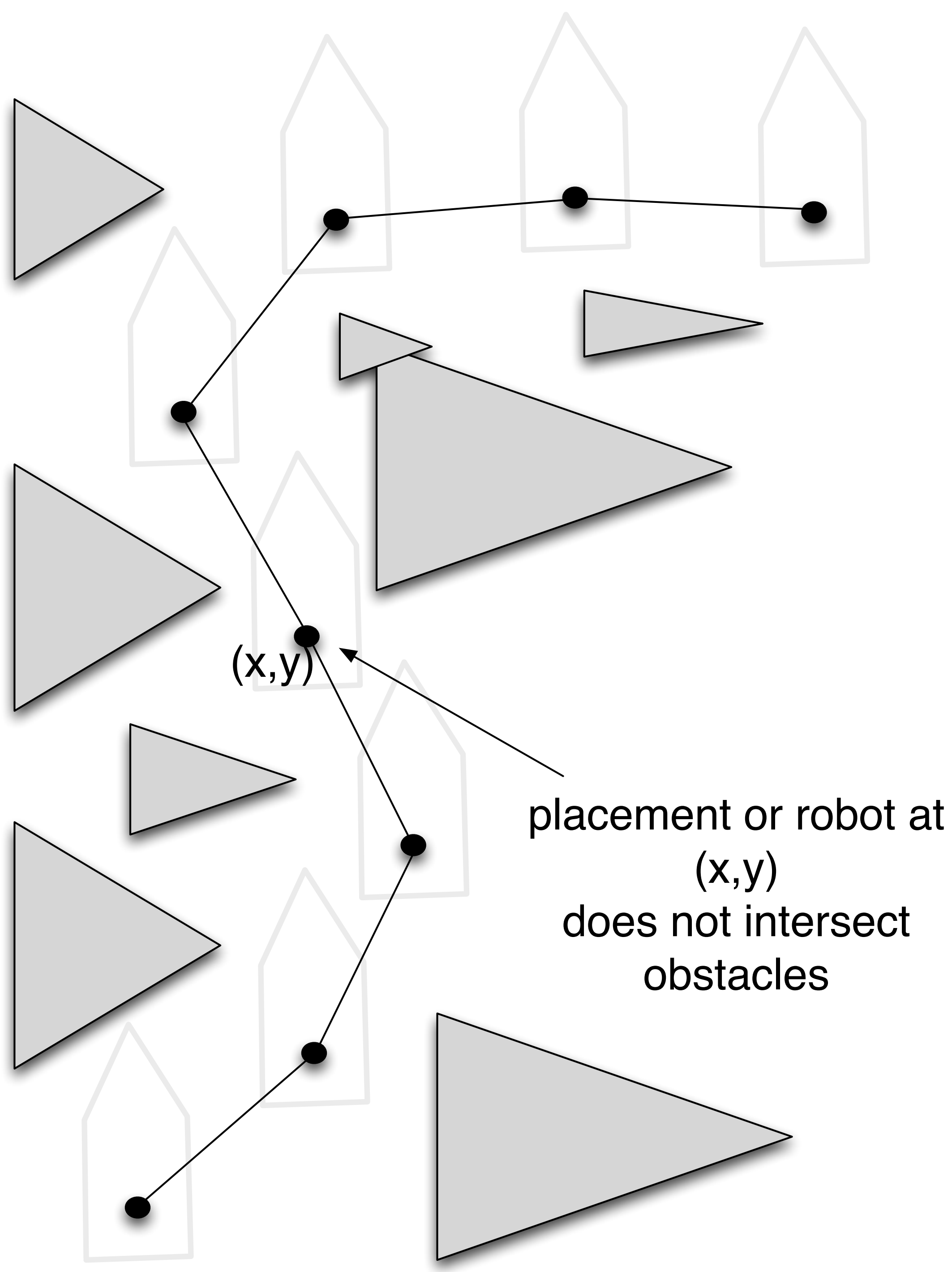
Motion planning in C-space

- Any path for R corresponds to a path for R in C-space
- We want a path that does not intersect obstacles



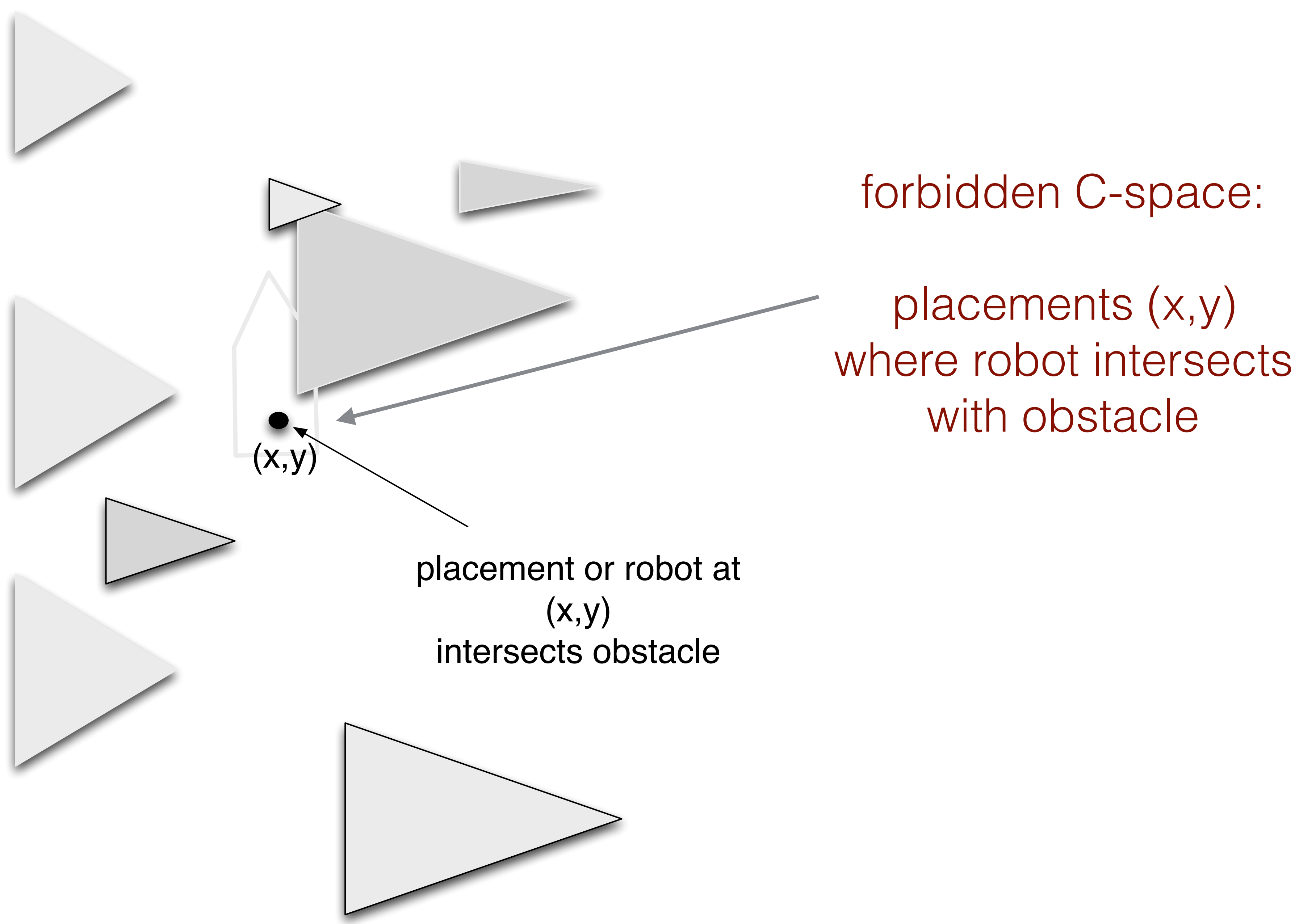
Motion planning in C-space

- Free C-space



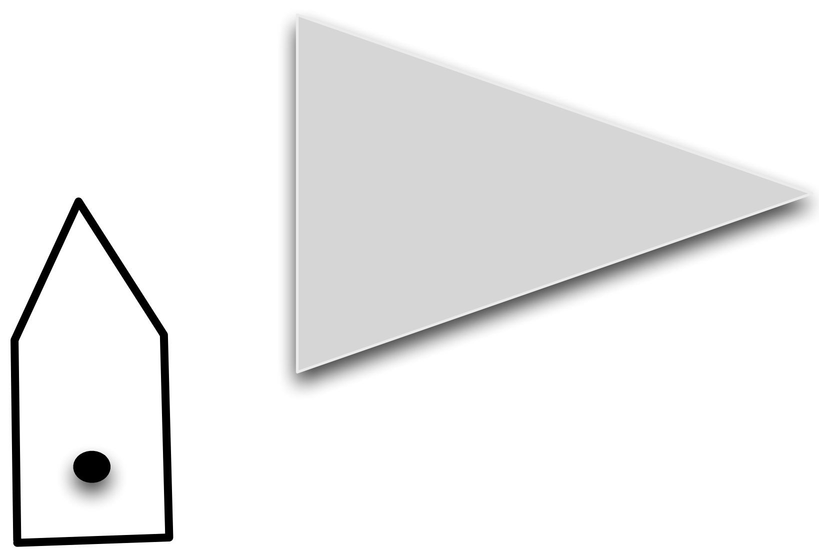
Motion planning in C-space

- Forbidden C-space



C-obstacles

- Extended obstacle, or C-obstacle
  - Given obstacle O, robot R(x,y): what placements cause intersection with O?



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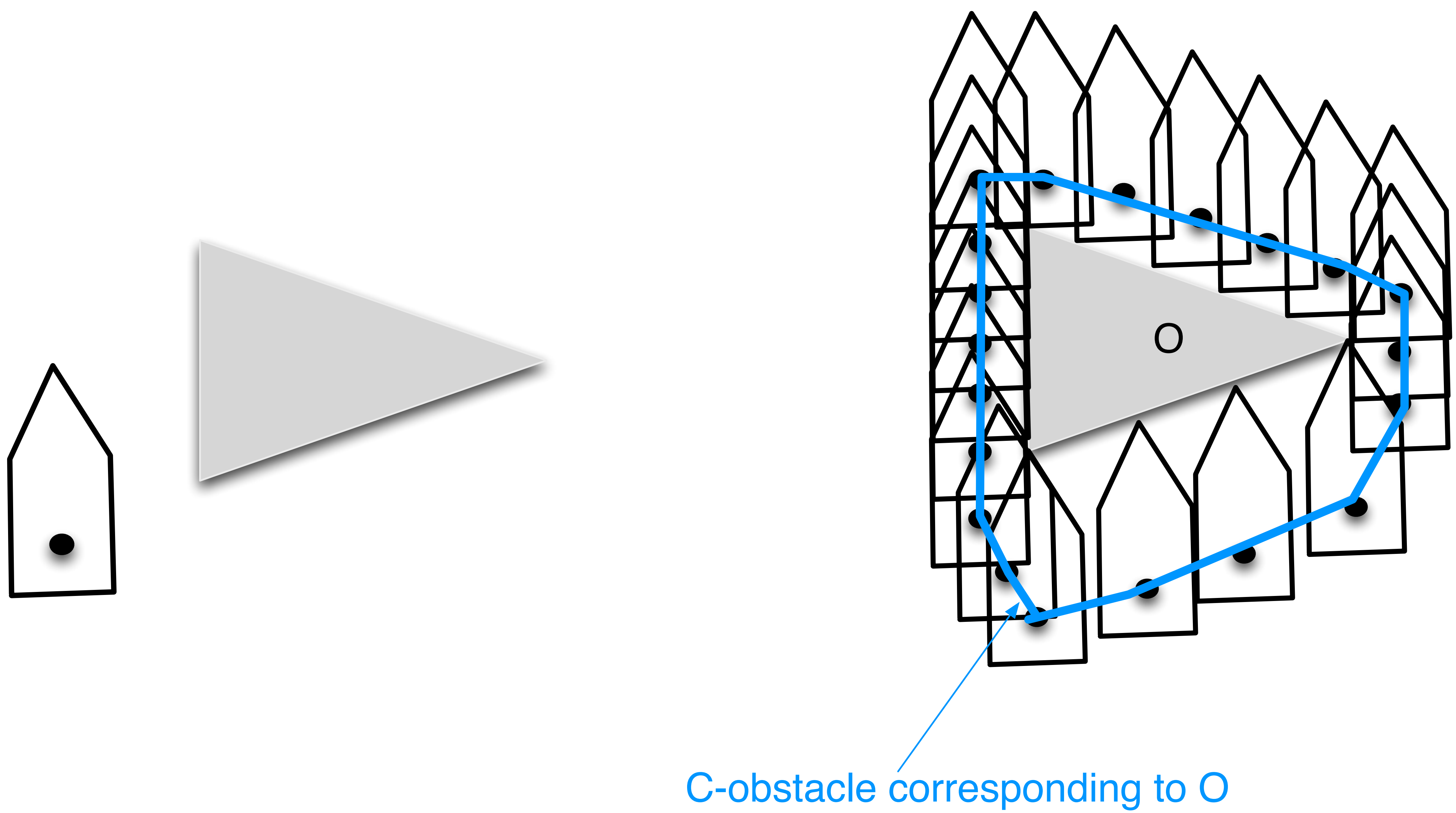

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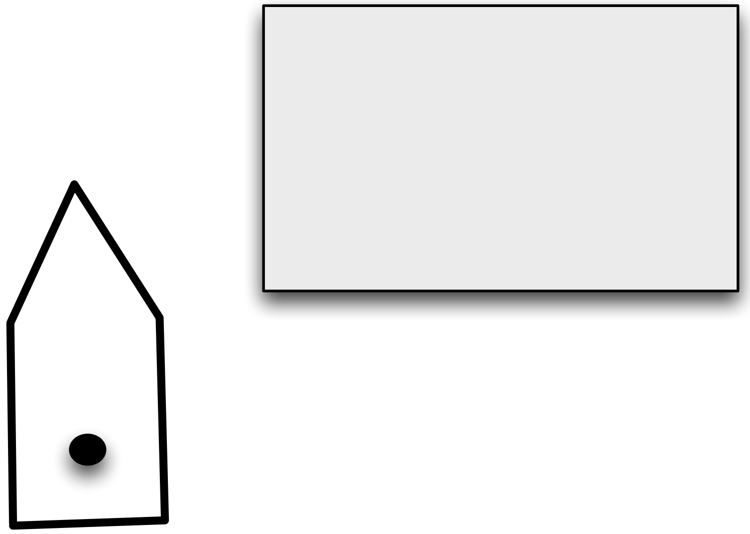

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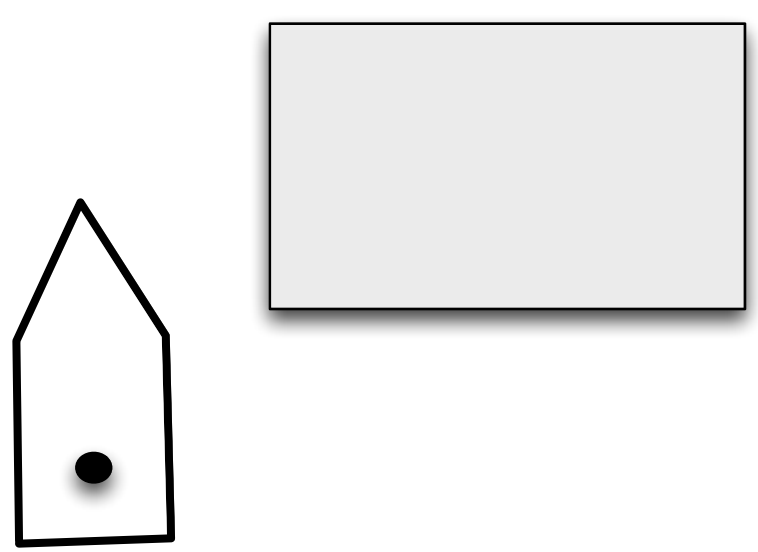
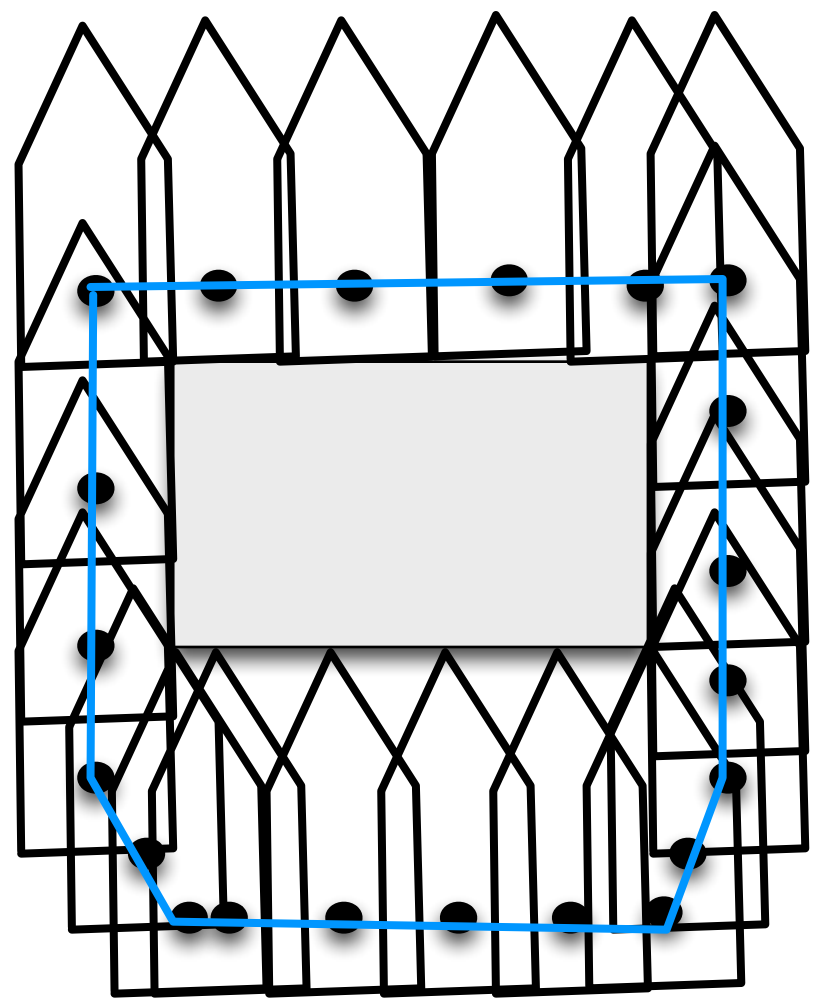


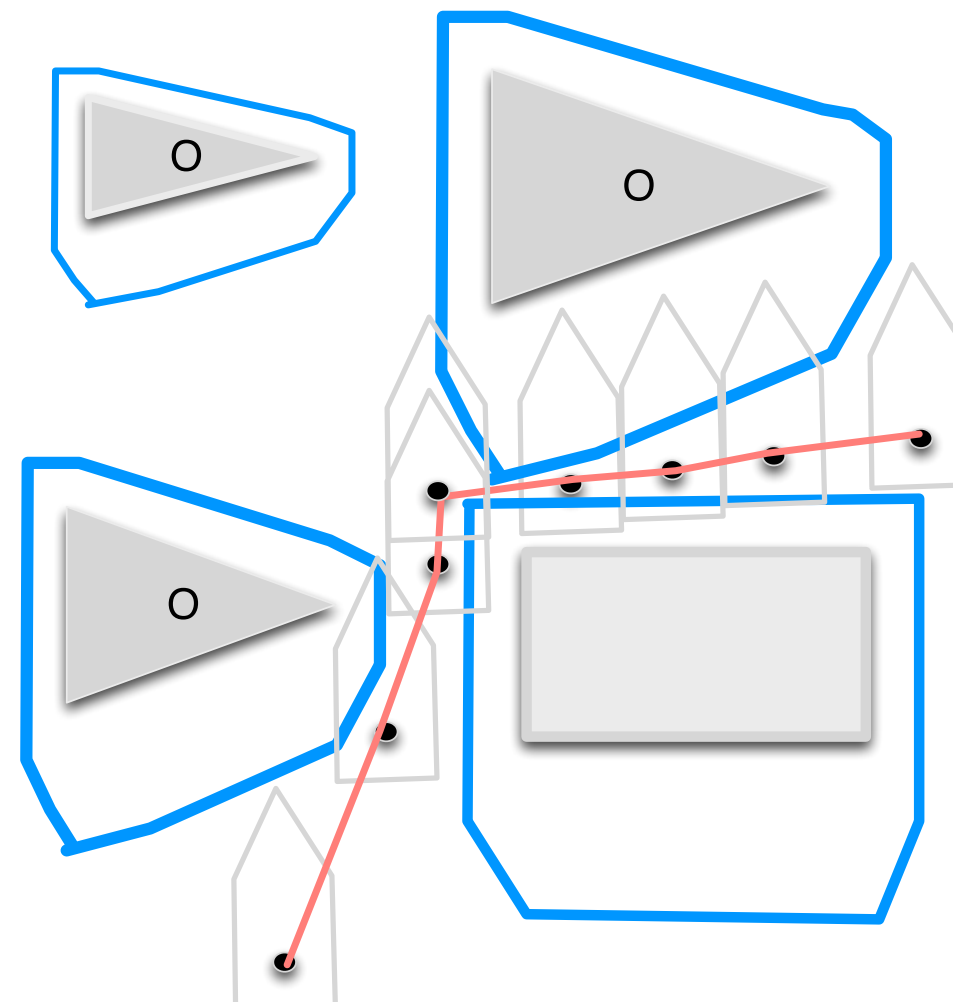


Translational motion planning

- Algorithm for polygonal robot translating in 2D
  - For each obstacle O, compute the corresponding C-obstacle
  - Compute the union of C-obstacles
  - Compute its complement. That's the free C-space.

  //now the problem is reduced to point robot moving in free C-space

  - Compute a trapezoidal map of free C-space
  - Compute a roadmap



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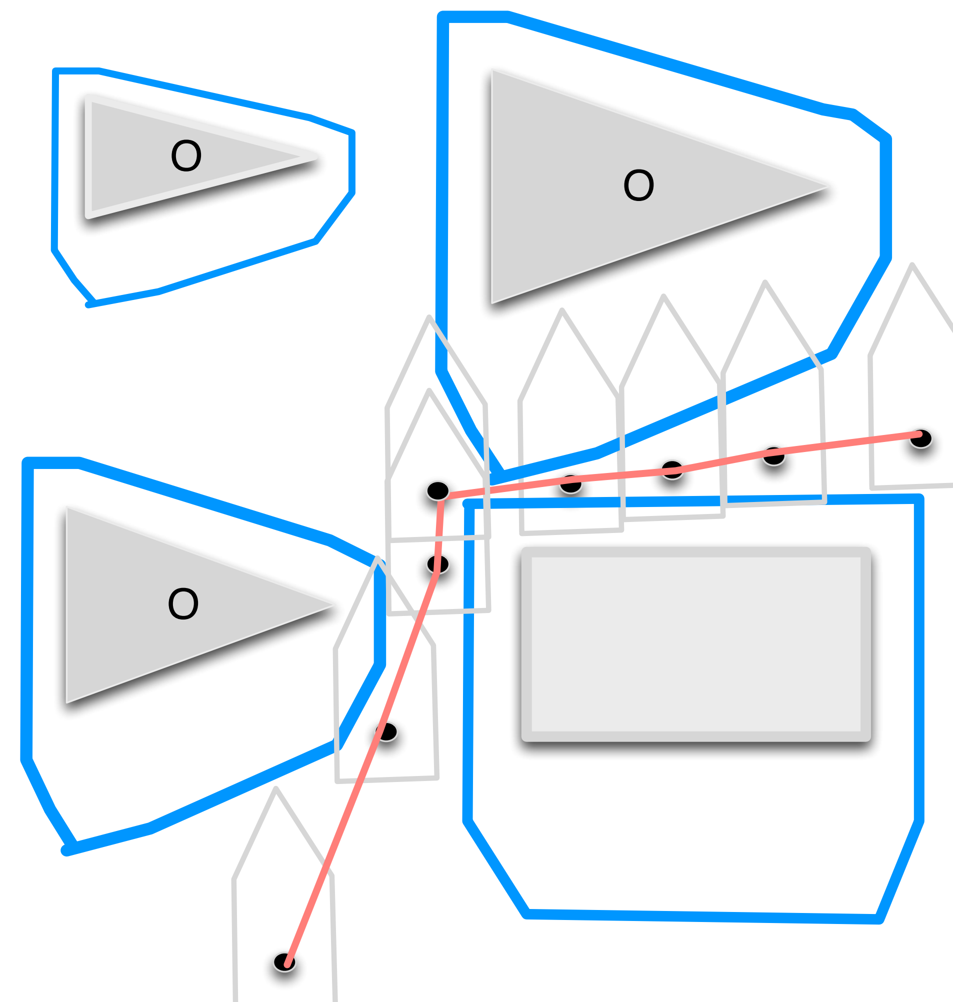

How long?
Is it complete? Optimal?

Path planning: A Preview of Approaches

Geometric/combinatorial

- Roadmaps
  - via trapezoidal decomposition of free space
  - Shortest path-roadmaps (Visibility graph)
  - max-clearance roadmaps (Voronoi diagrams)

Pros:

- complete, optimal
- elegant

Cons:

- practical only in simple instance (disks, convex shapes, <= 3 DOF)

Approximate/Heuristical

- approximate cell decomposition
- potential methods
- incremental sampling
- probabilistic roadmaps

Pros:

- work in many realistic scenarios and finds paths

Cons:

- not complete, not optimal
- often no efficiency guarantees