# Computational Geometry

[csci 3250]

Laura Toma

Bowdoin College

# The Art Gallery Problem

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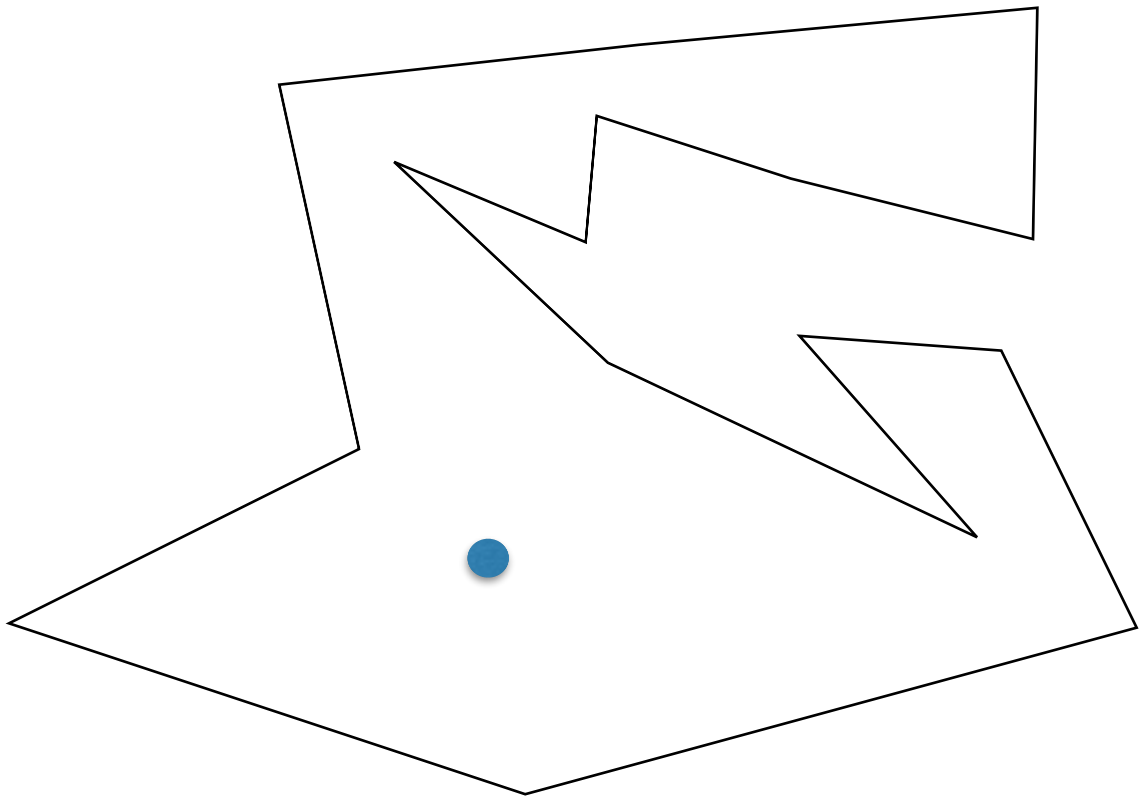
Imagine an art gallery whose floor plan is a simple polygon, and a guard (a point) inside the gallery.



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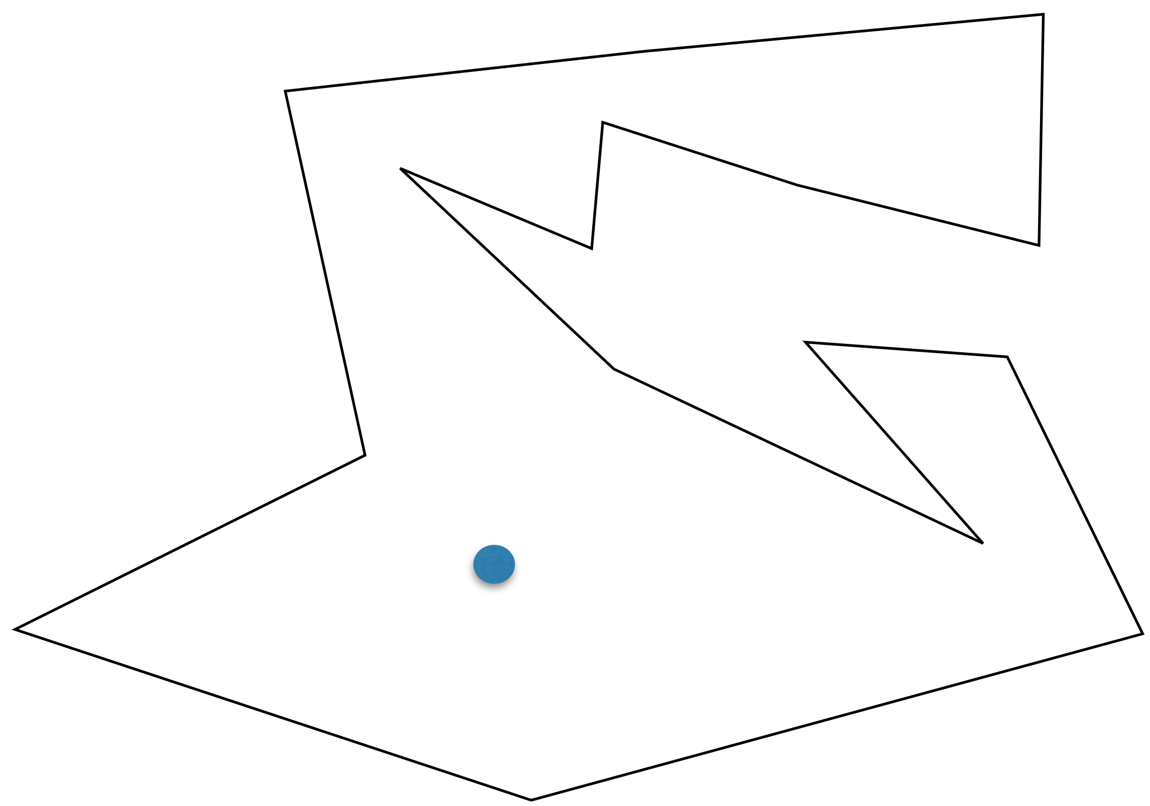

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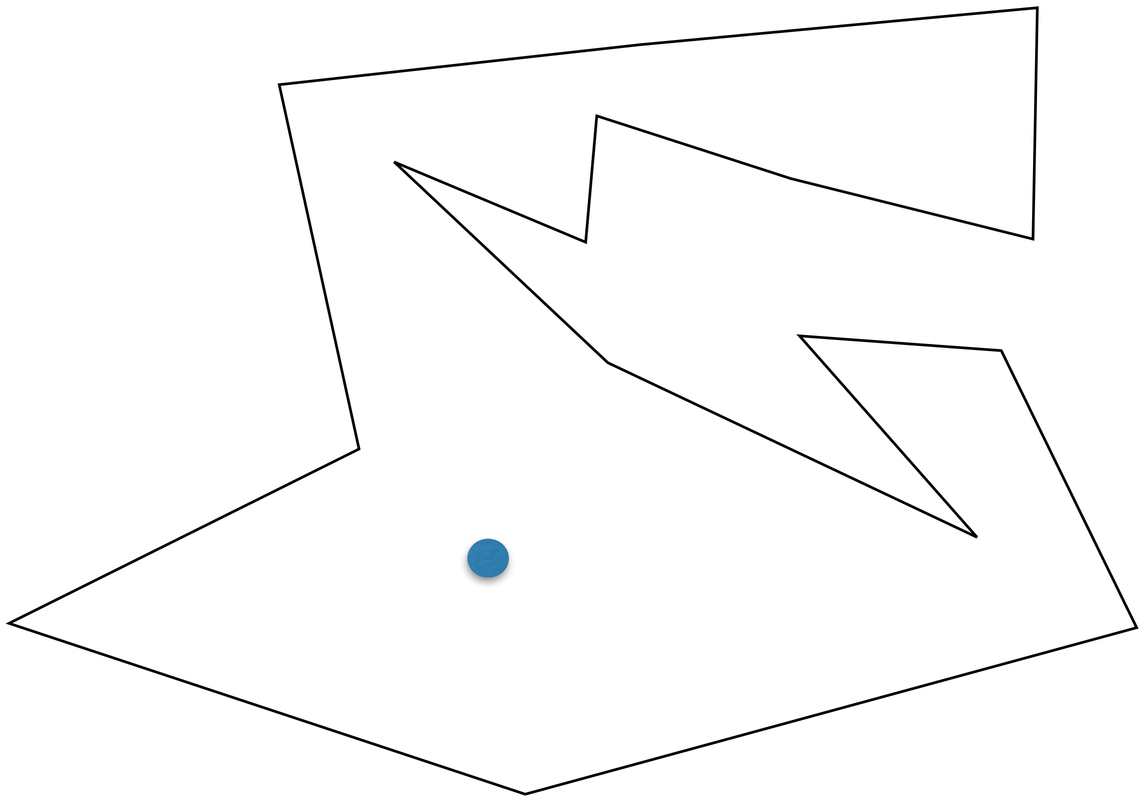

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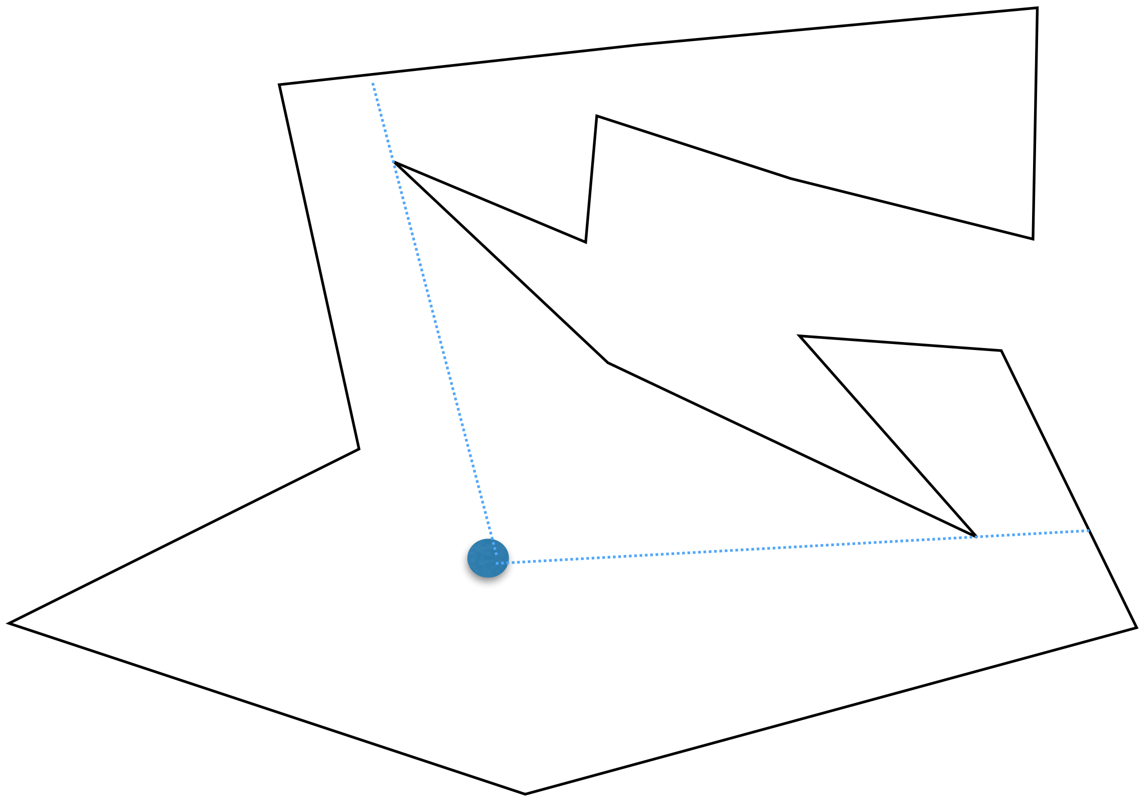

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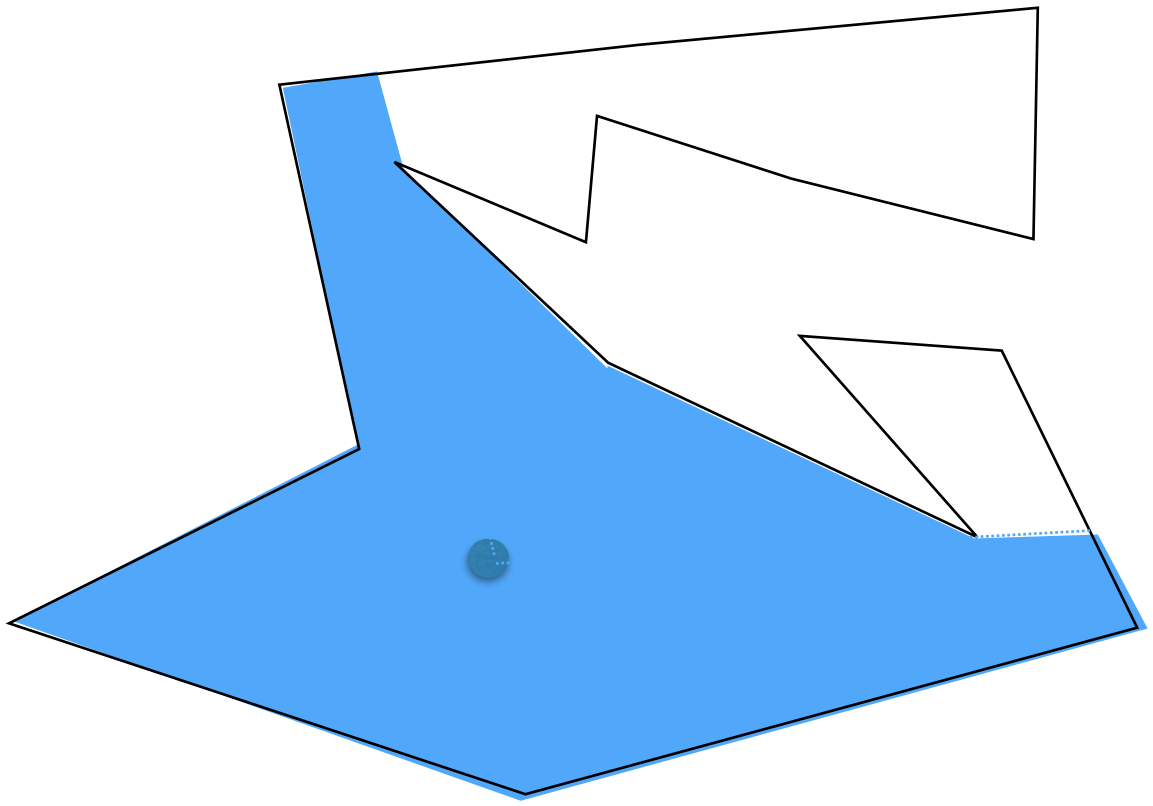
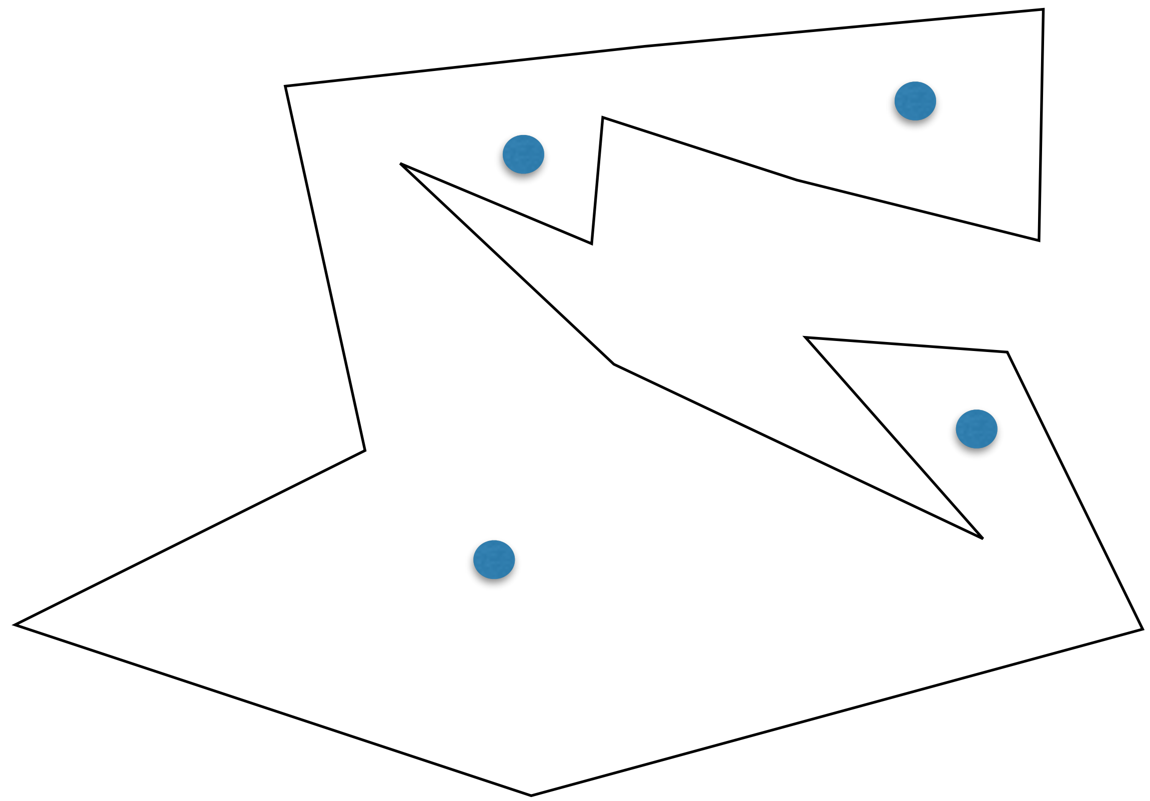


# The Art Gallery Problem

We say that a set of guards **covers** P if every point in P is visible to al test one guard.

Questions:

- Given P, what is the smallest number of guards (and their locations) to cover P?
  - NP-complete
- Klee's problem: Given a polygon of n vertices, what is the minimum number of guards to cover the polygon? Find the maximum over all polygons of size n.



# Klee's Problem

Notation

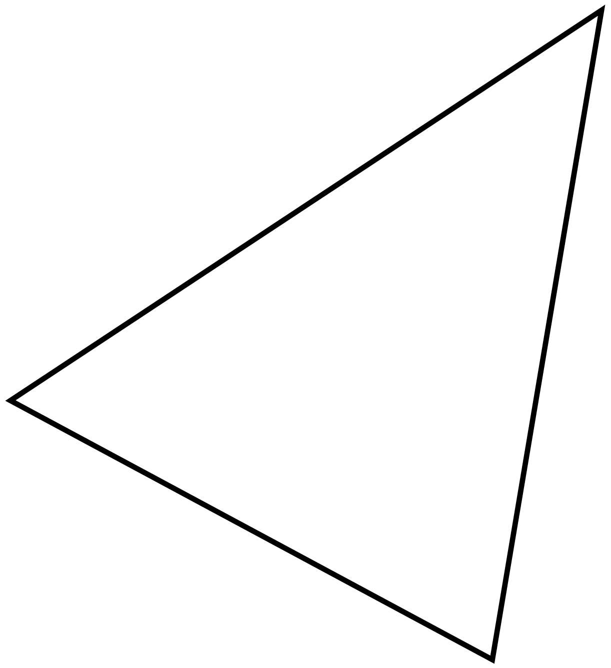
- $P_n$: polygon of n vertices
- g(P) = the smallest number of guards to cover P
- let $G(n) = \max \{ g(P_n) \mid \text{all } P_n \}$
- G(n) is the smallest number of guards necessary to guard a polygon of n vertices

- Klee's problem: find G(n)
- Note
  - G(n) is necessary: there exists a $P_n$ that requires G(n) guards
  - G(n) is sufficient: any $P_n$ can be guarded with G(n) guards

- Trivial bounds
  - $G(n) >= 1$
  - $G(n) <= n$ (place one guard in each vertex)

# Klee's Problem

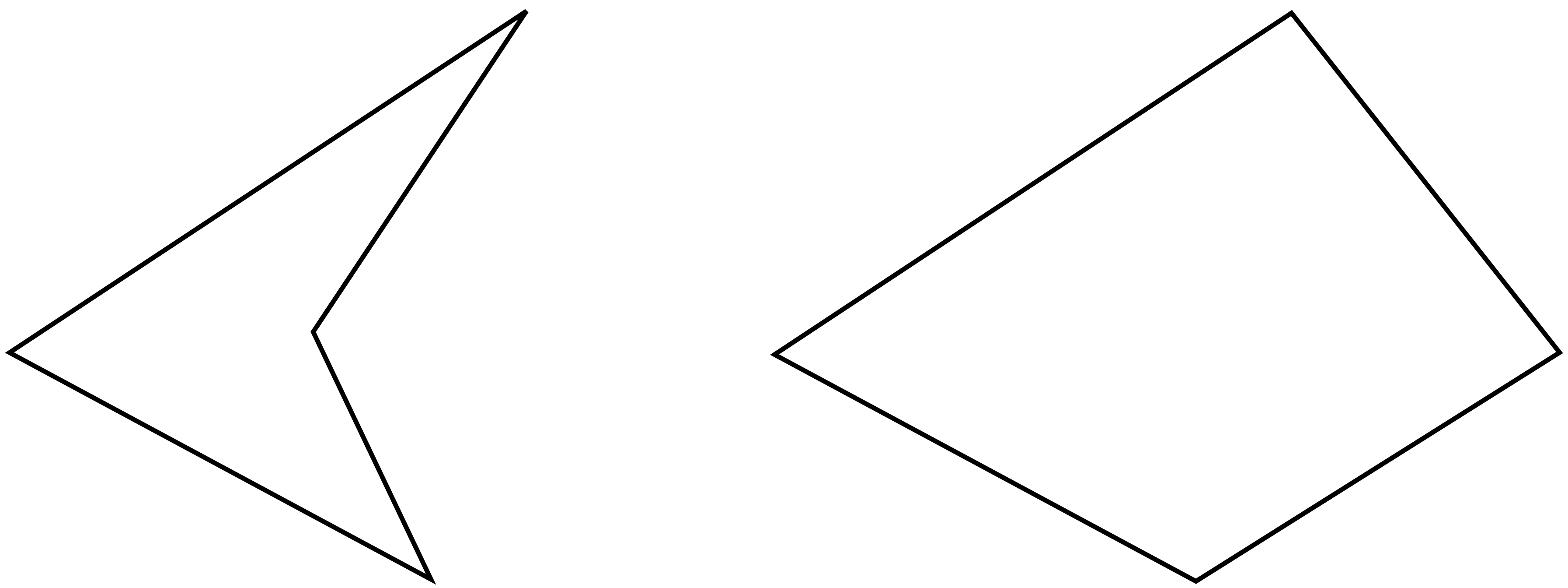
n=3



G(3) = 1

# Klee's Problem

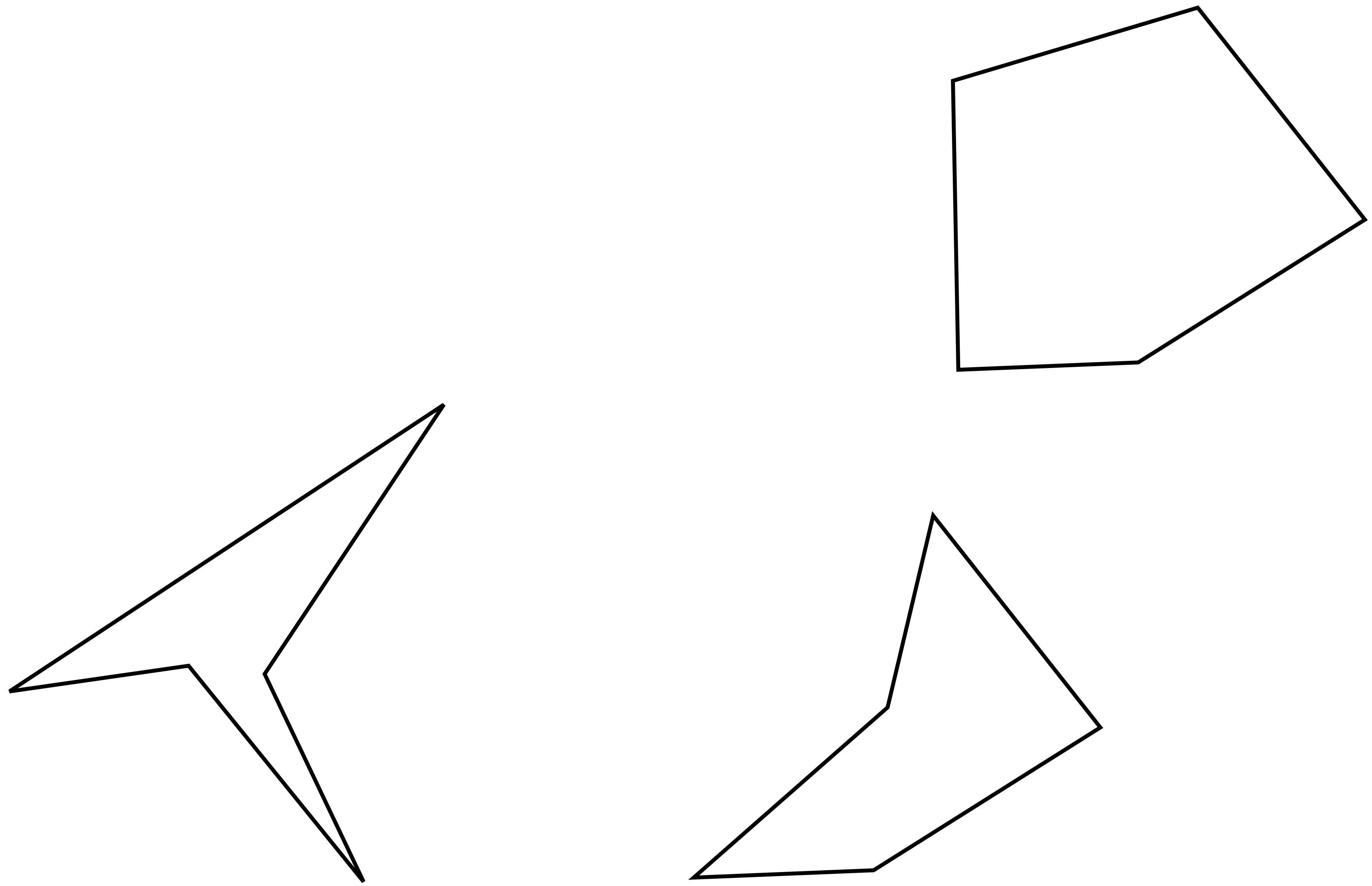
n=4



G(4) = 1

# Klee's Problem

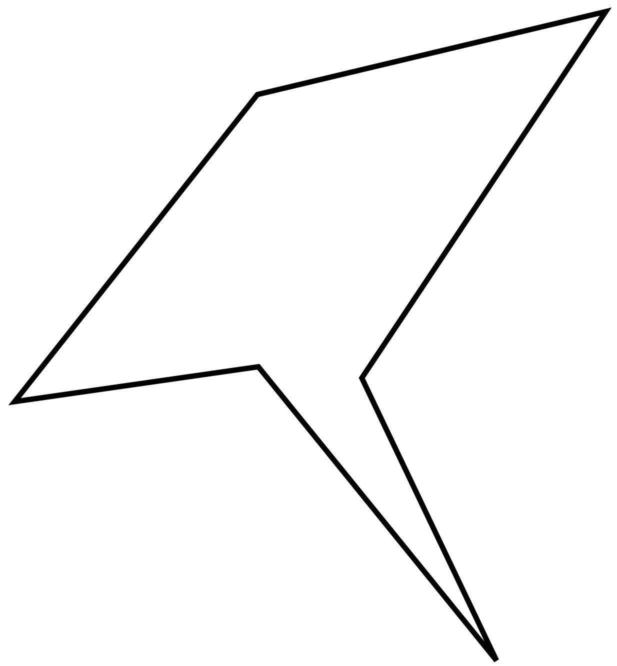
n=5

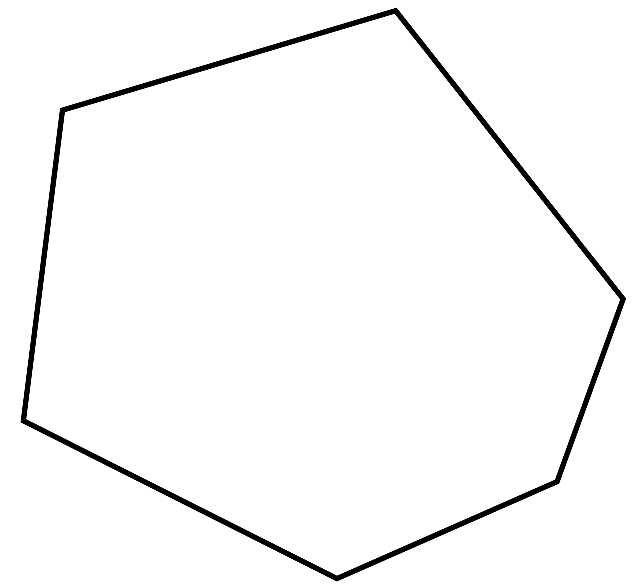


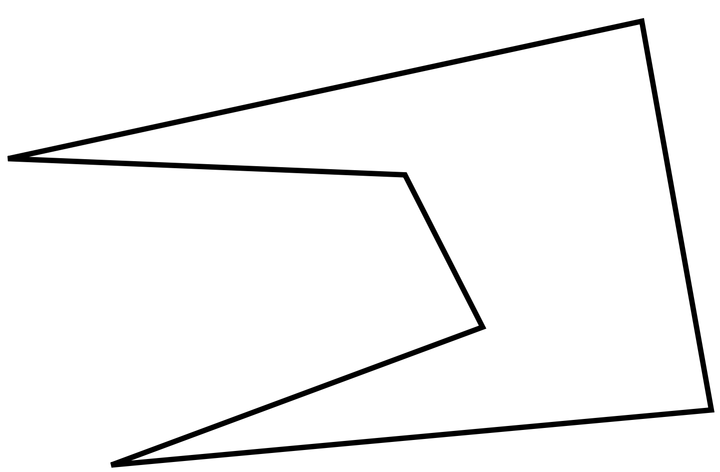
$G(5) = 1$

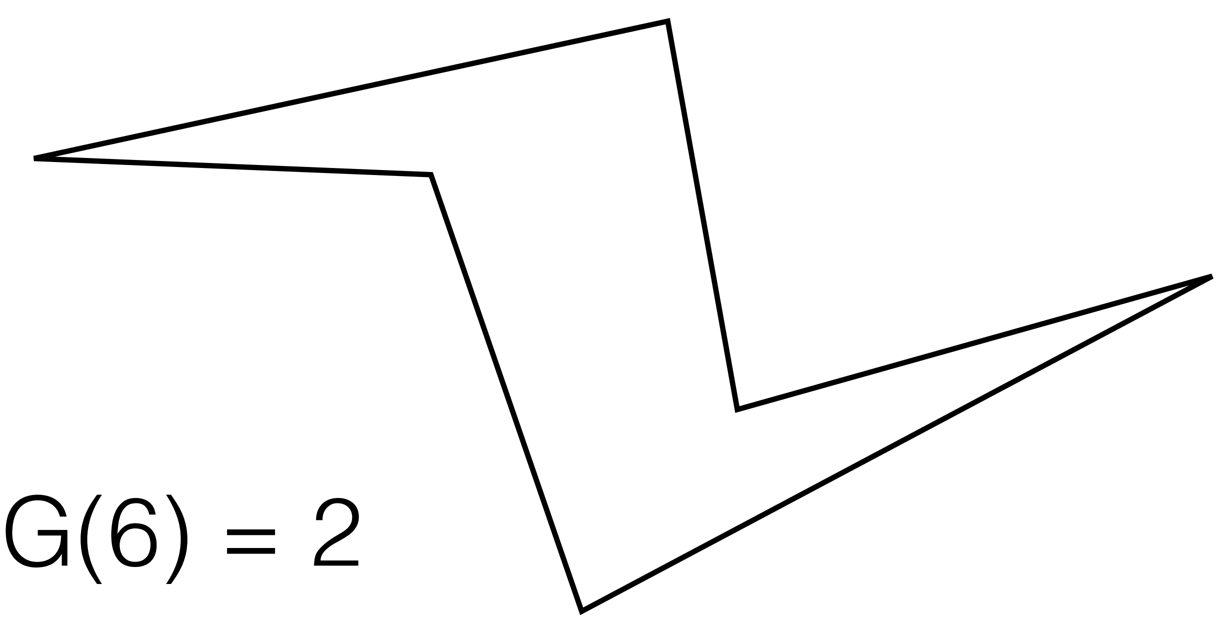
# Klee's Problem

n=6









G(6) = 2

# Klee's Problem

$G(n) = ?$

Come up with a $P_n$ that requires as many guards as possible.

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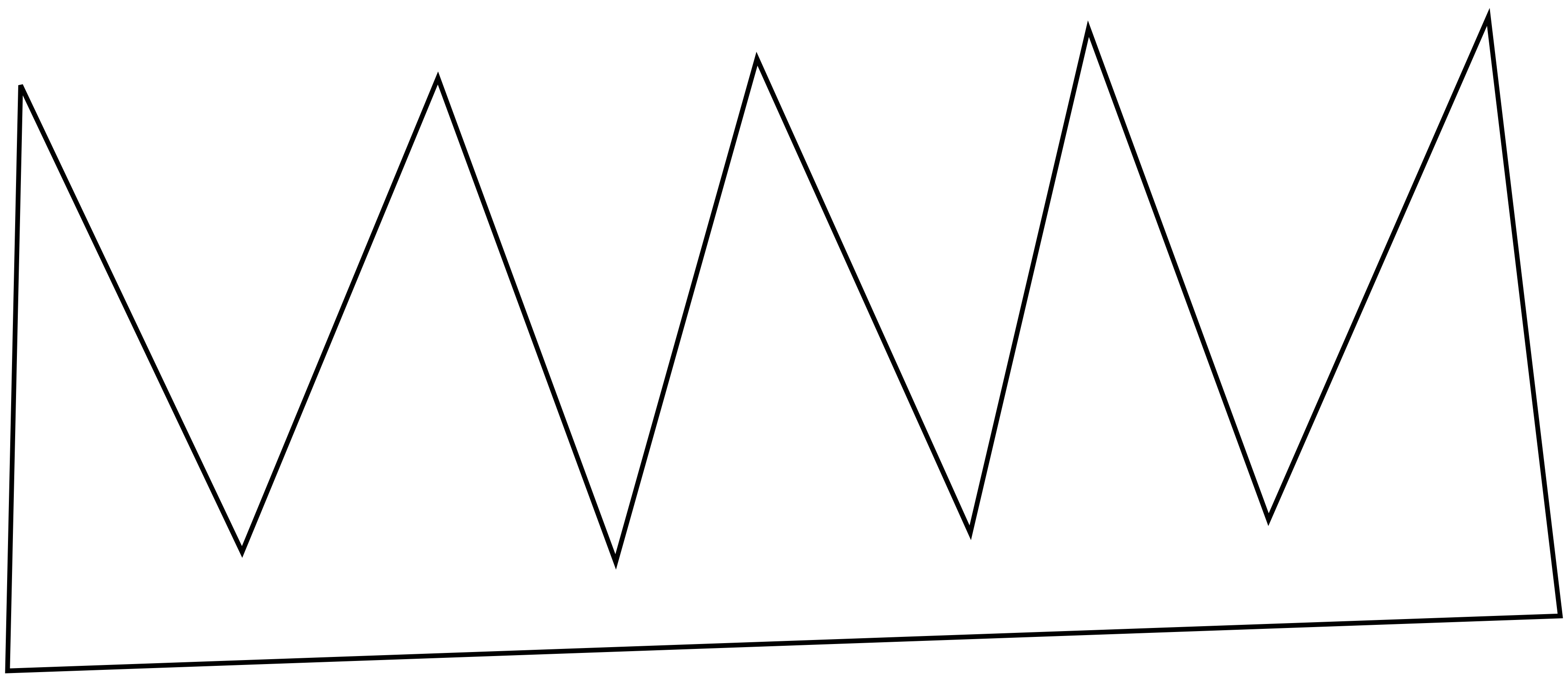

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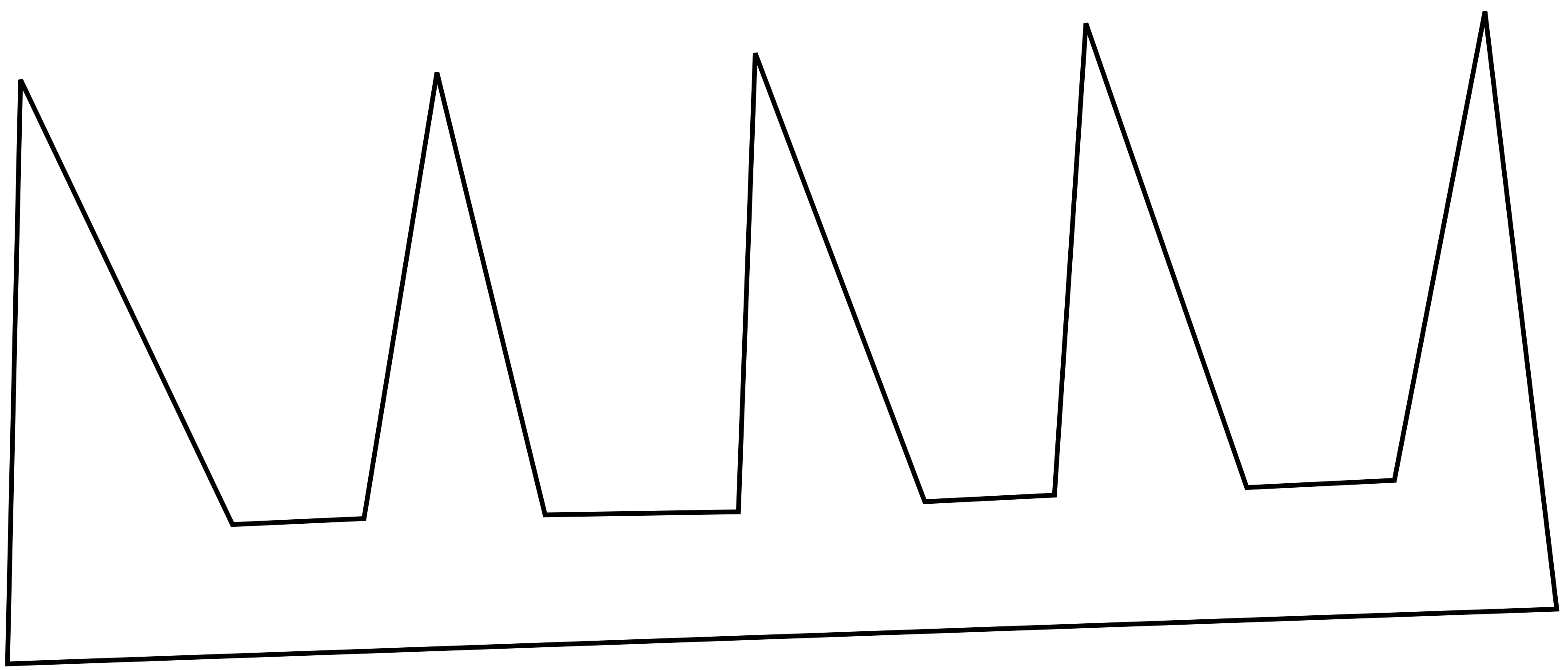
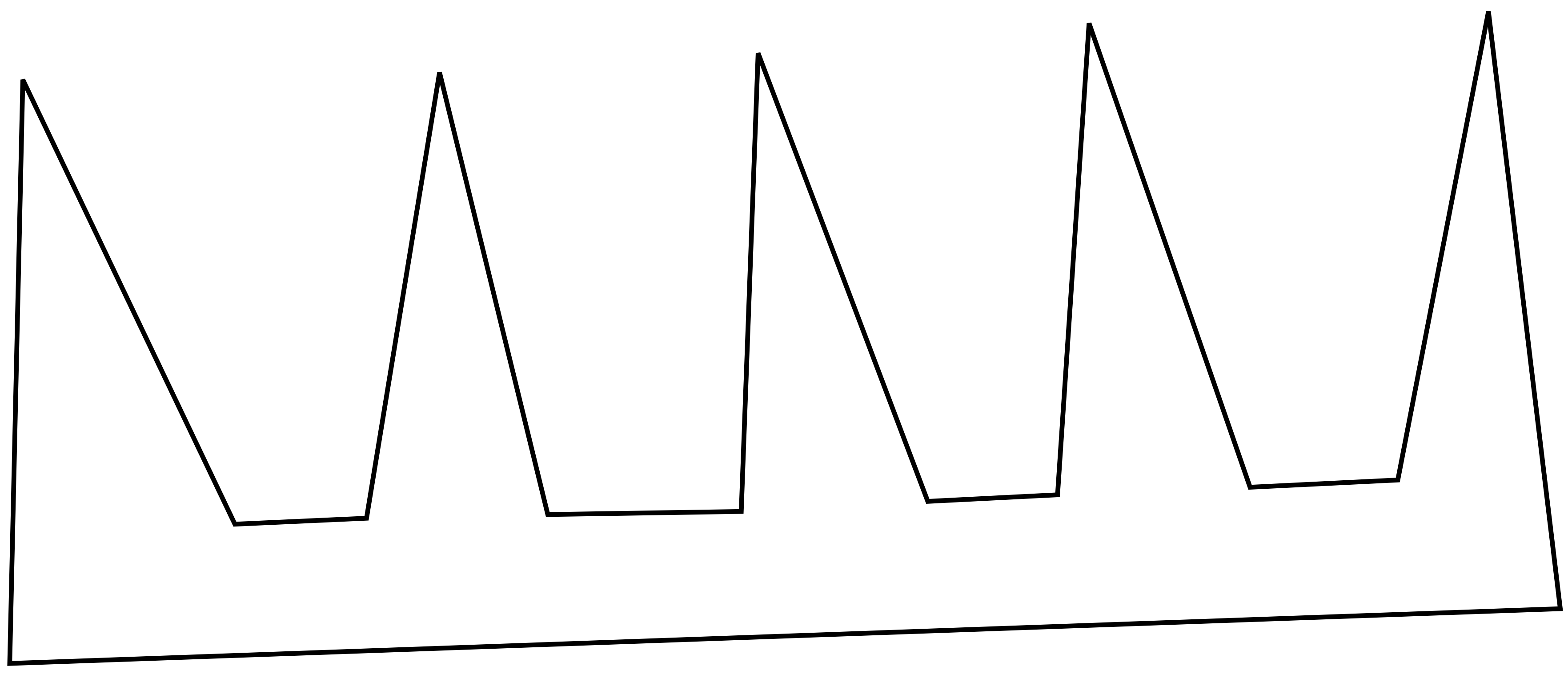


# Klee's Problem

$\lfloor n/3 \rfloor$ necessary



# Klee's Problem

It was shown that $\lfloor n/3 \rfloor$ is also sufficient:

that is, any Pn can be guarded with at most $\lfloor n/3 \rfloor$ guards

- (Complex) proof by induction
- Simple and beautiful proof due to Fisk (Bowdoin Math faculty)

# Fisk's proof of sufficiency

1. Any polygon can be triangulated
2. Any triangulation can be 3-colored
3. Observe that placing the guards at all the vertices assigned to one color guarantees the polygon is covered.
4. There must exist a color that's used at most n/3 times. Pick that color and place guards at the vertices of that color.

# Fisk's proof of sufficiency

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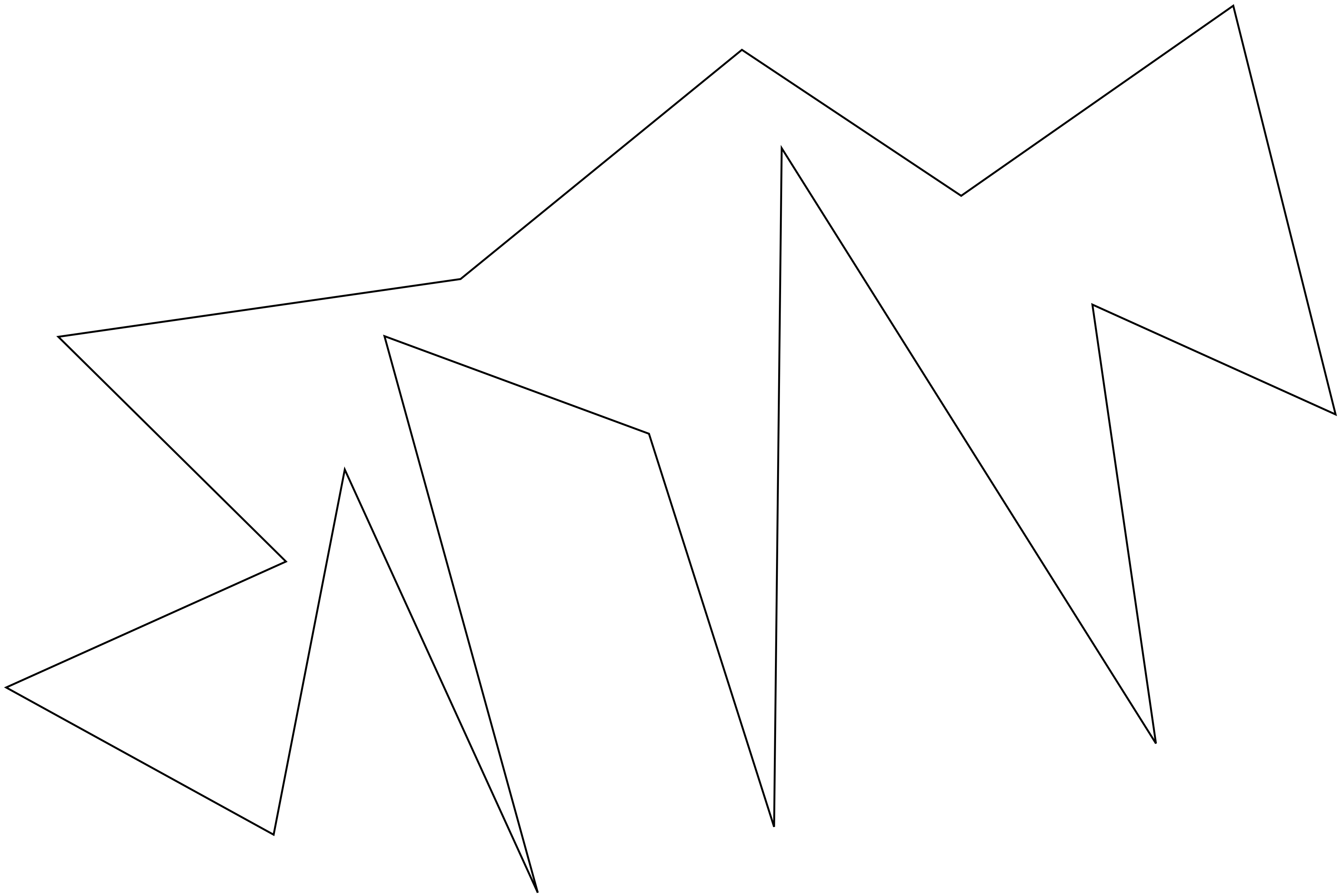

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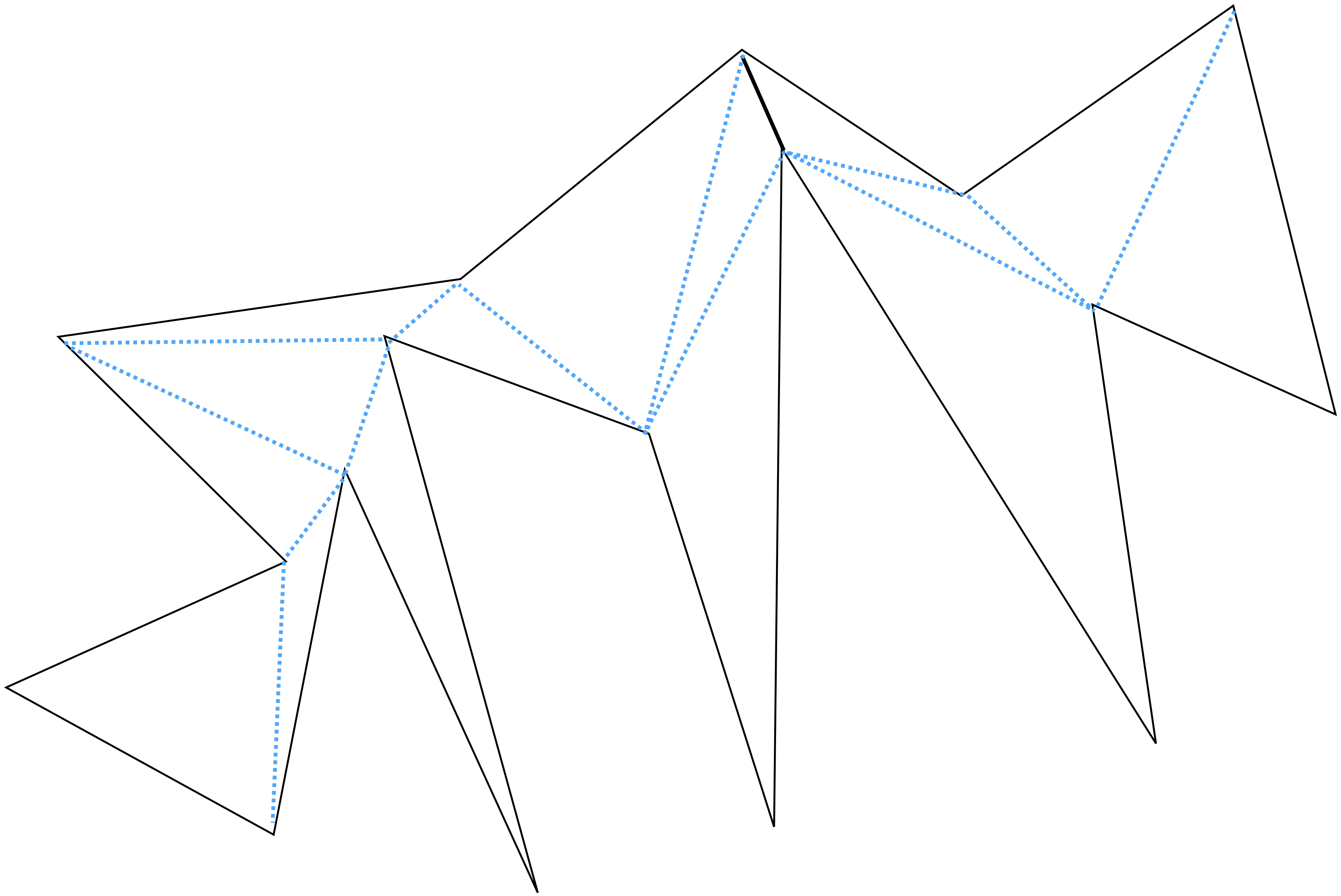

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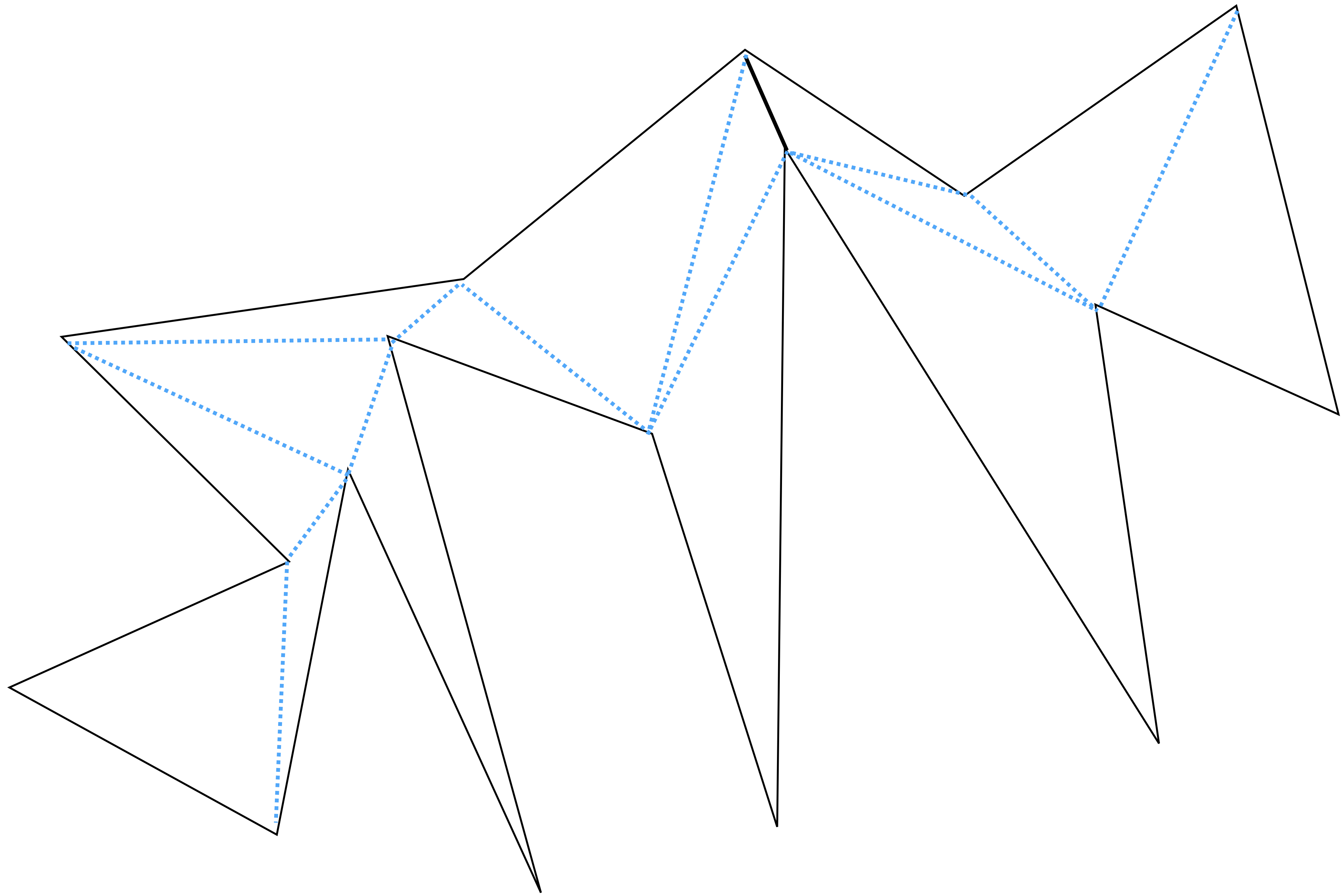

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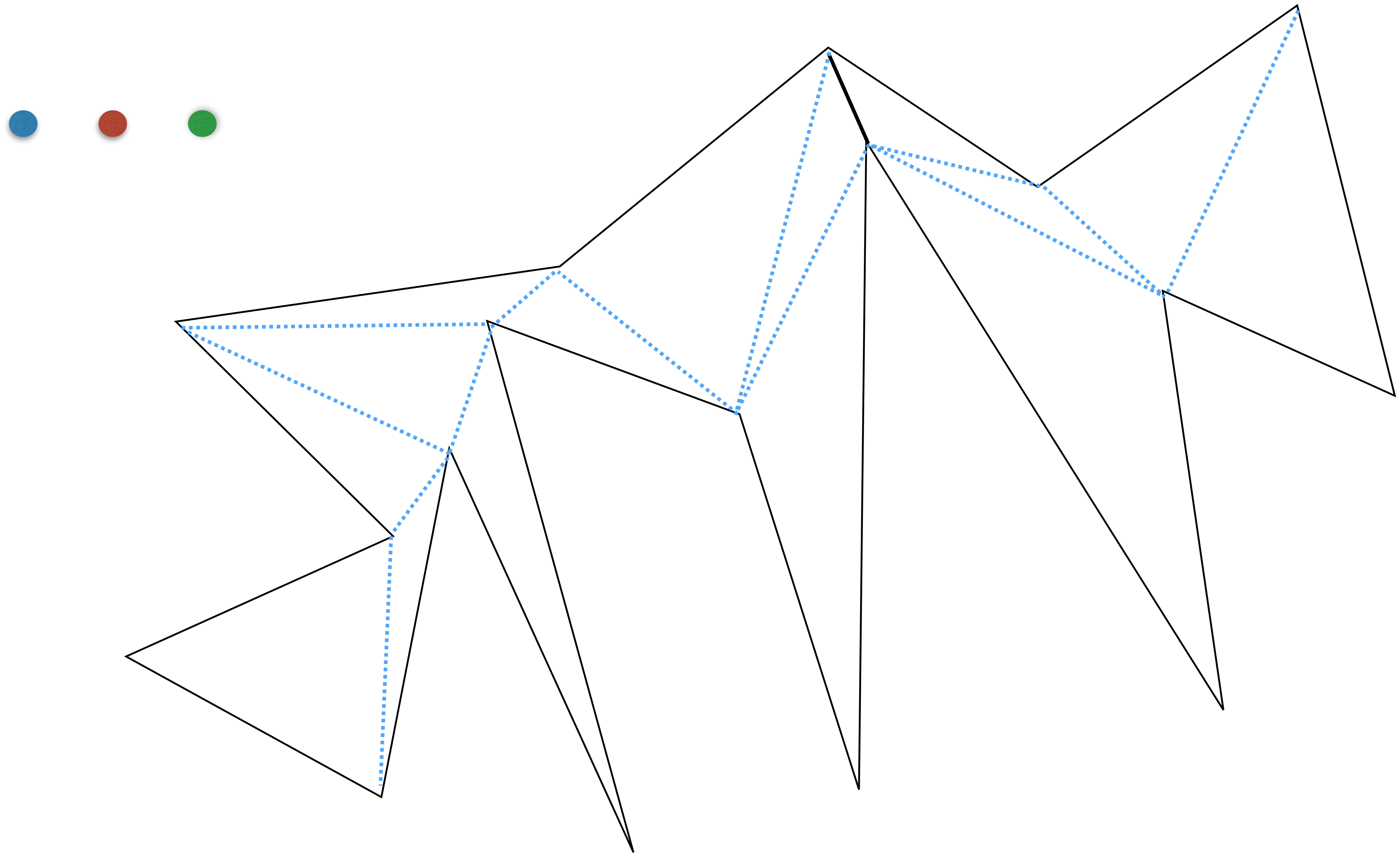

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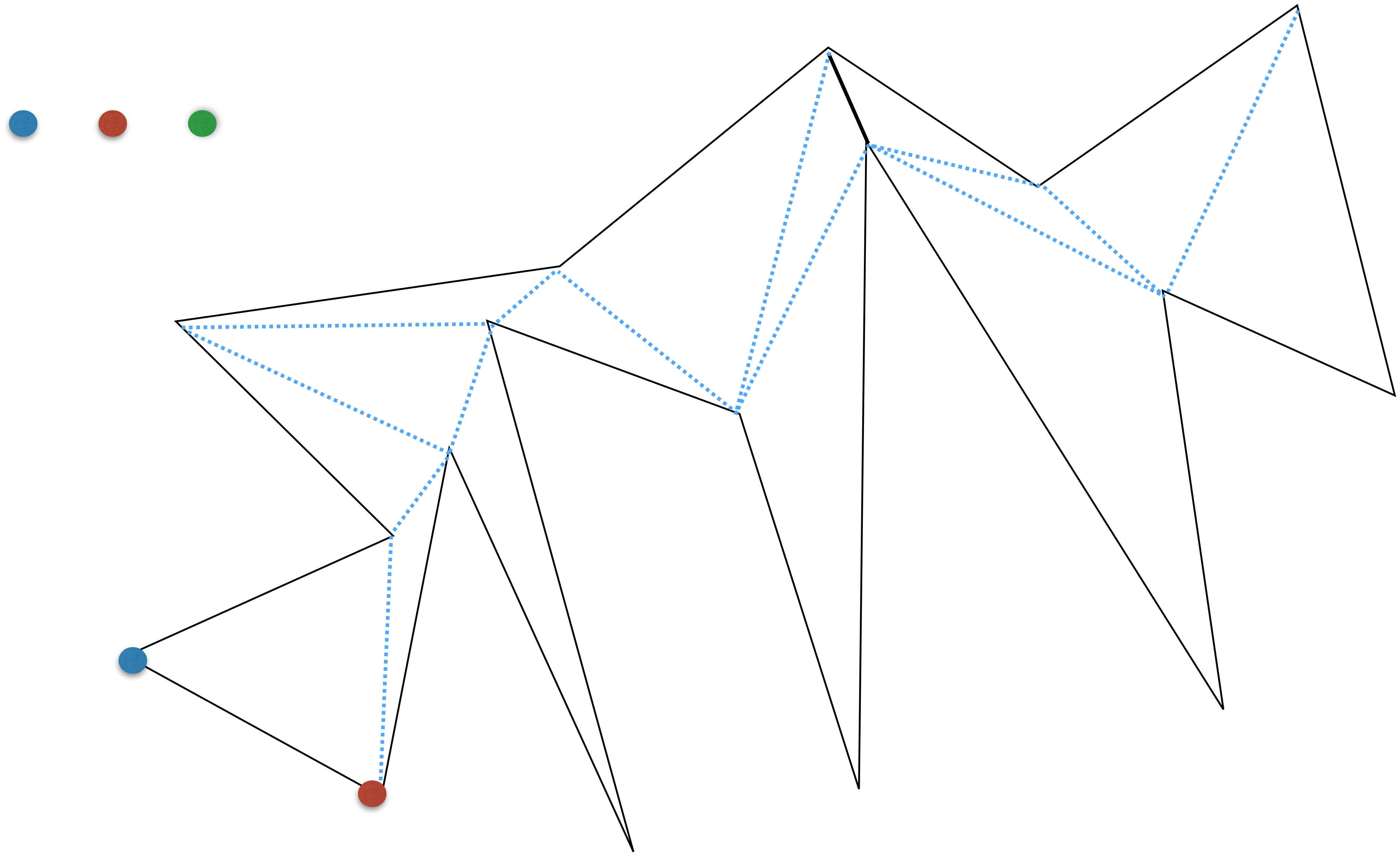

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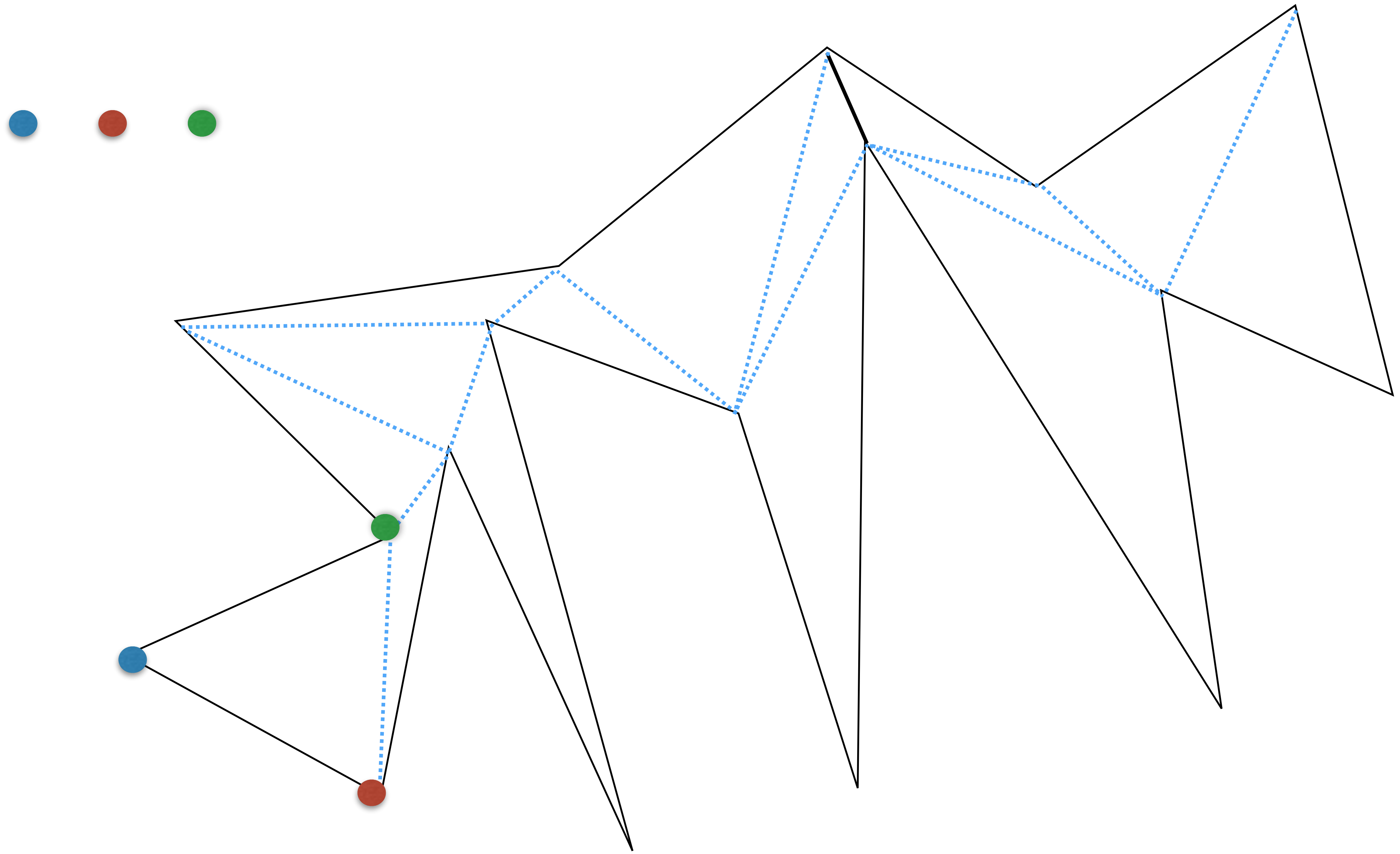

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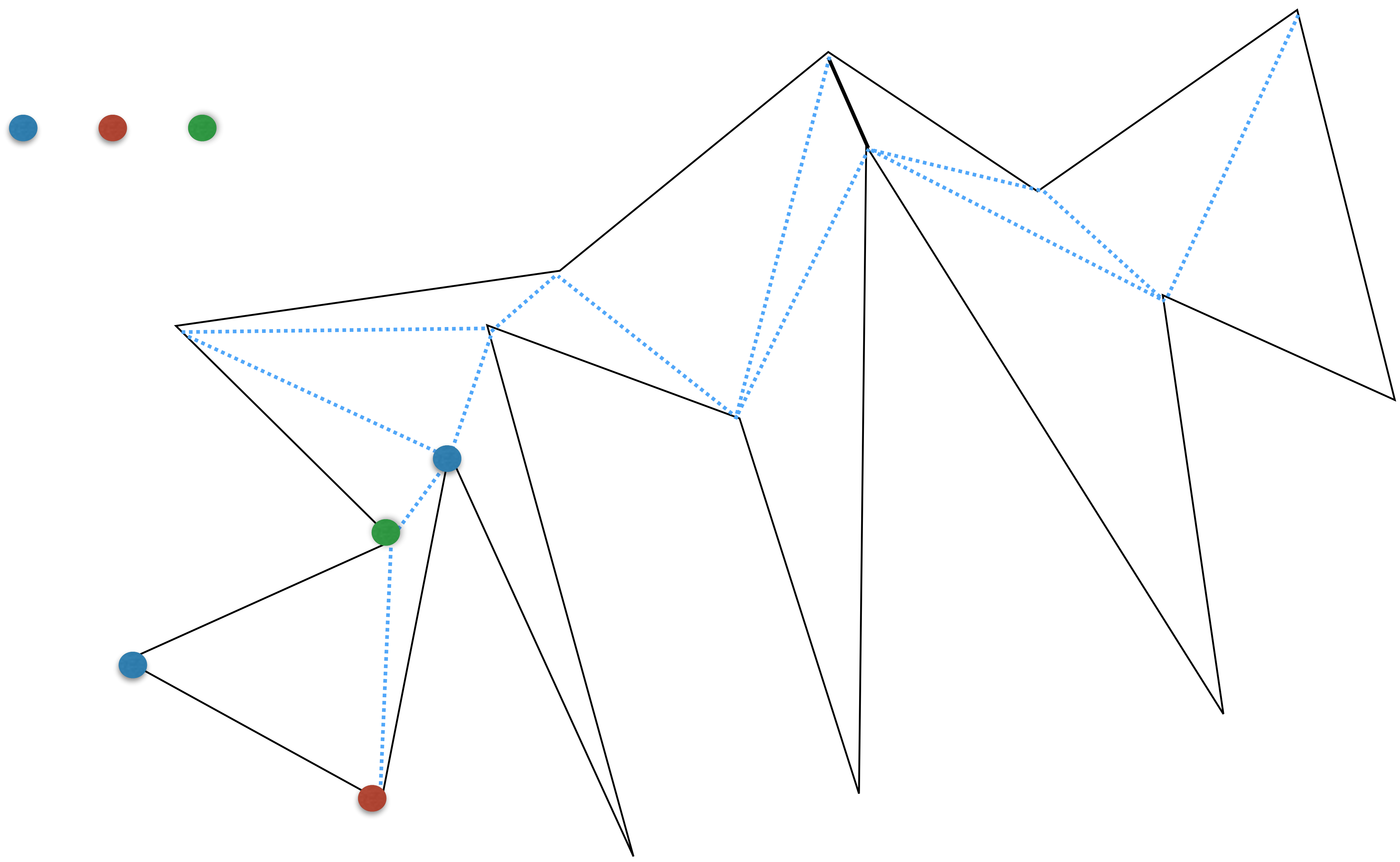

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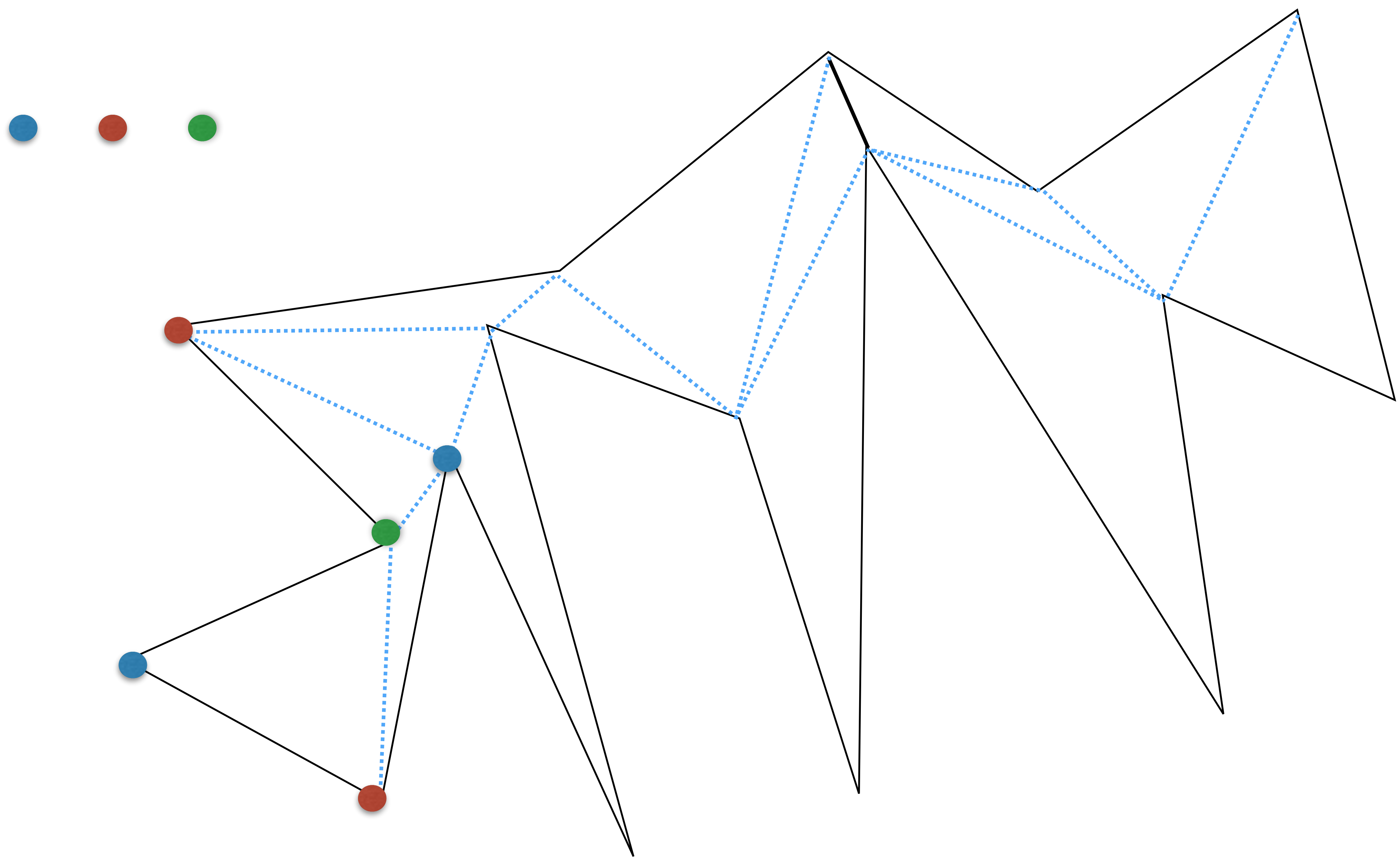

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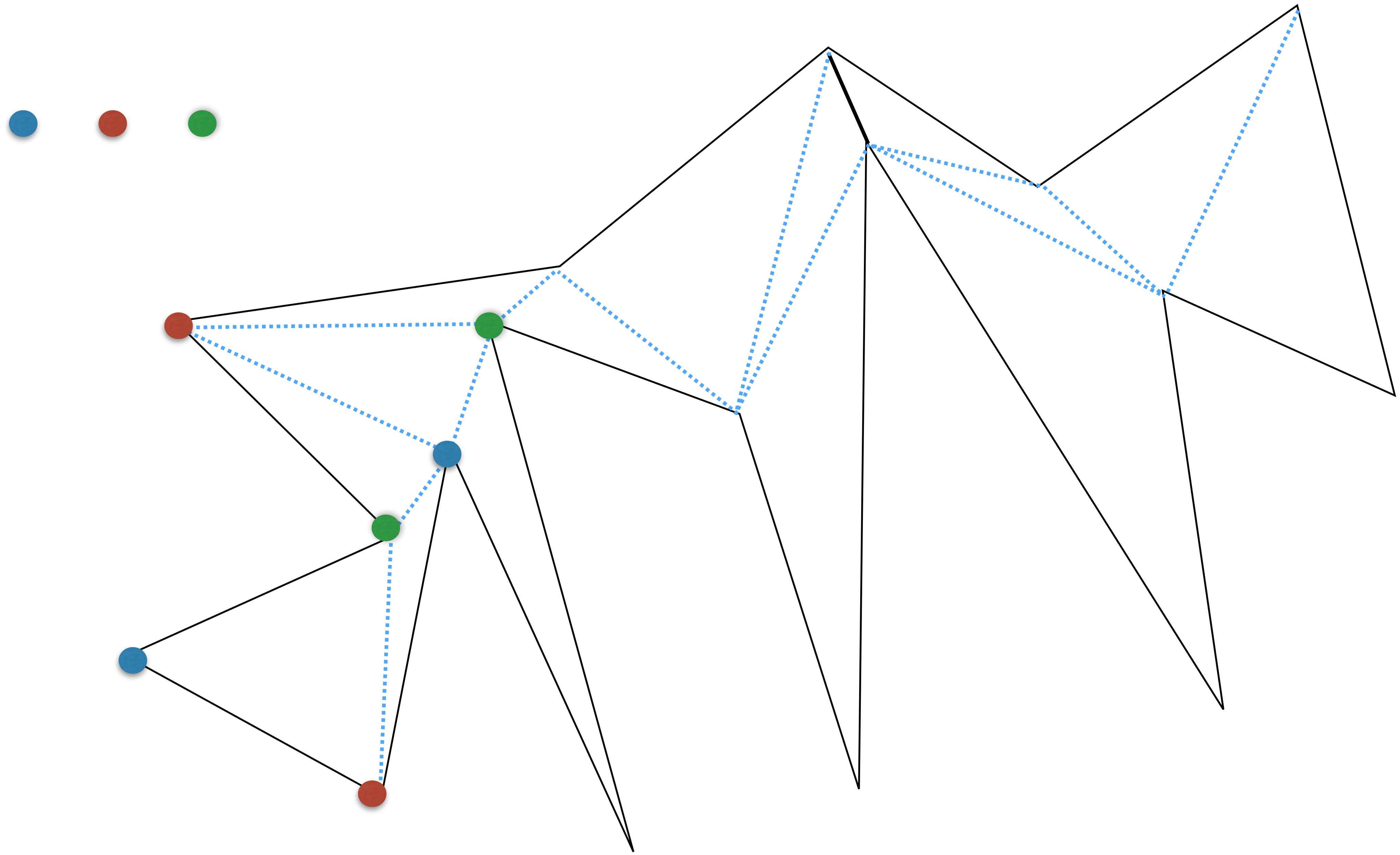

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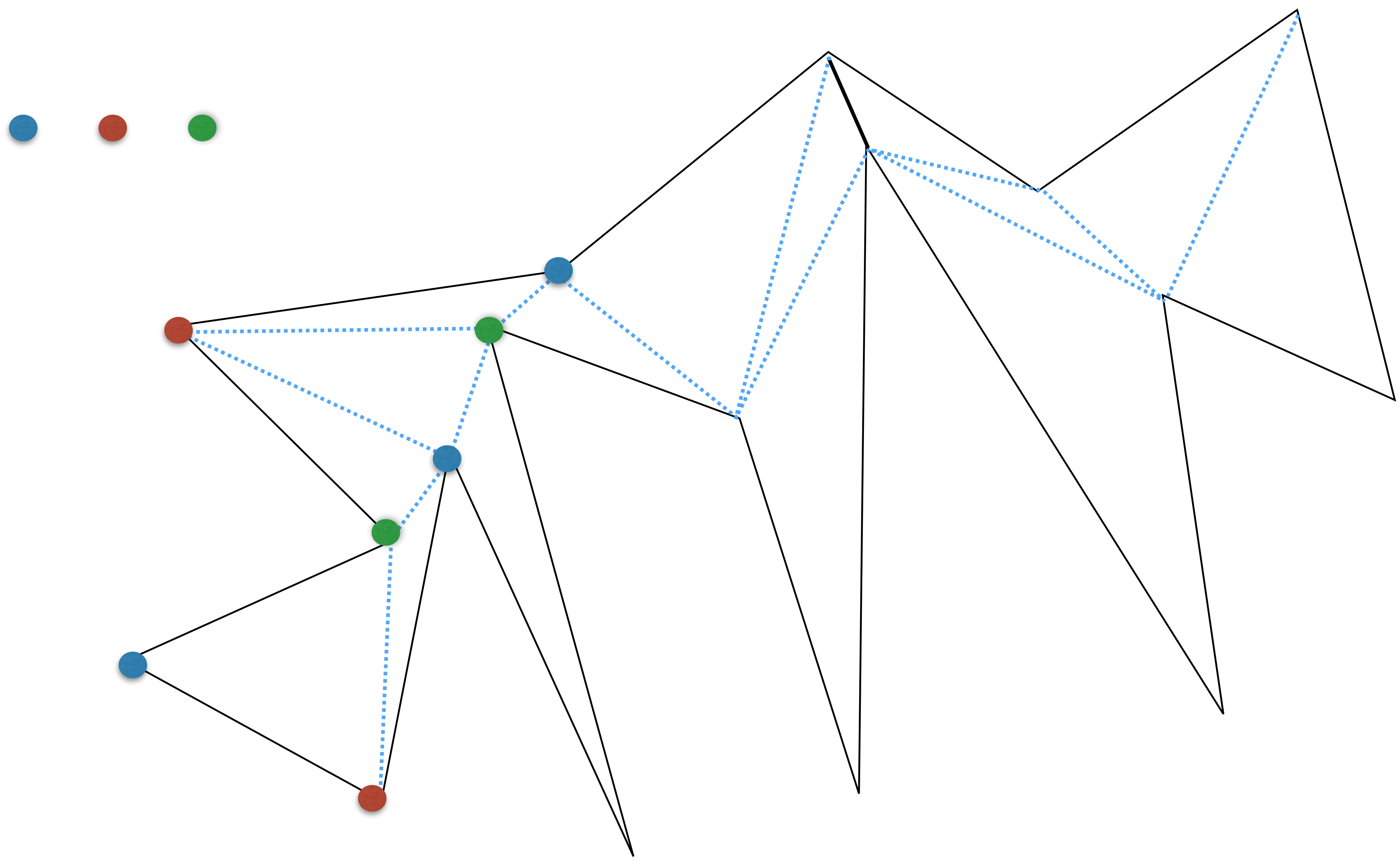

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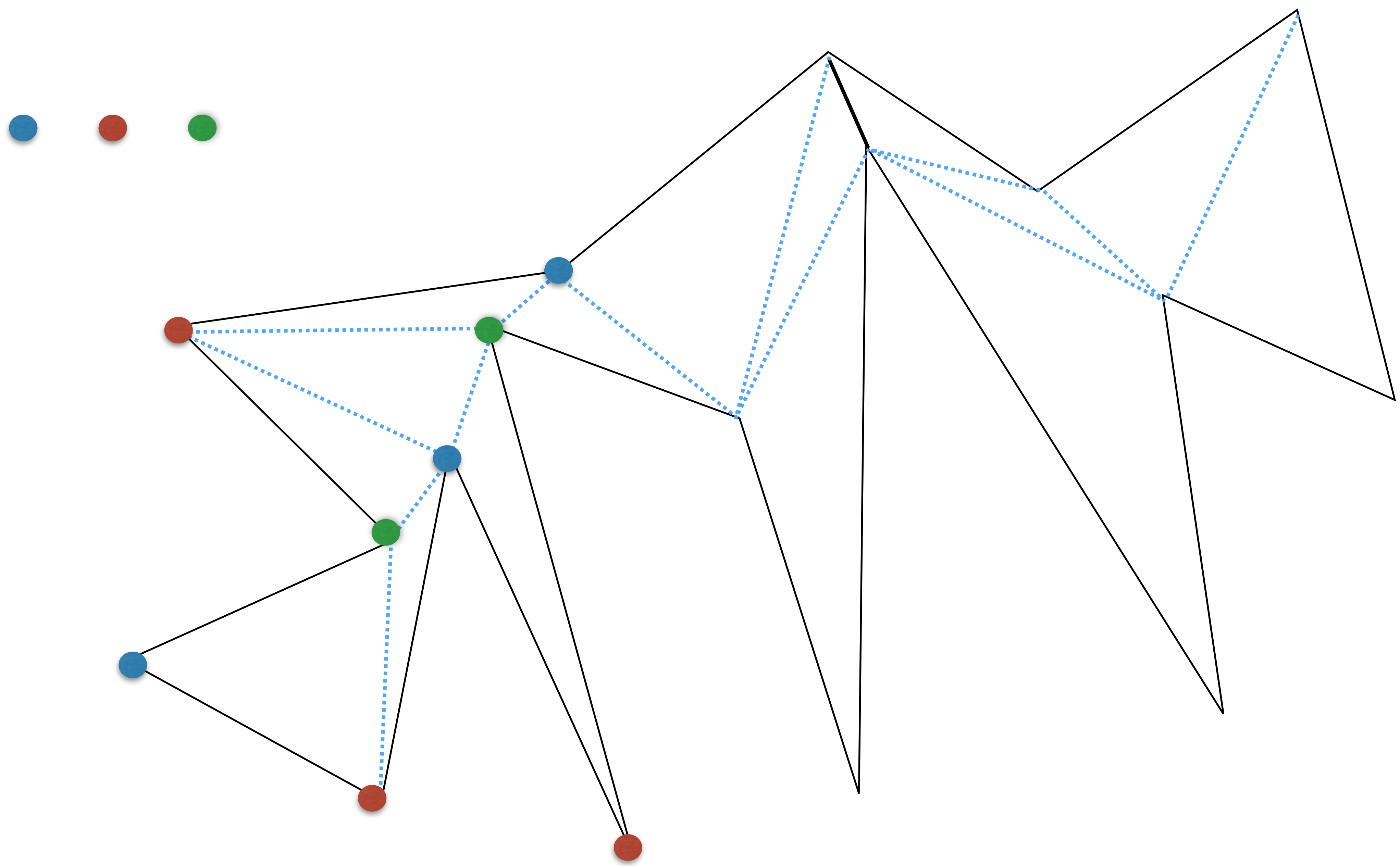

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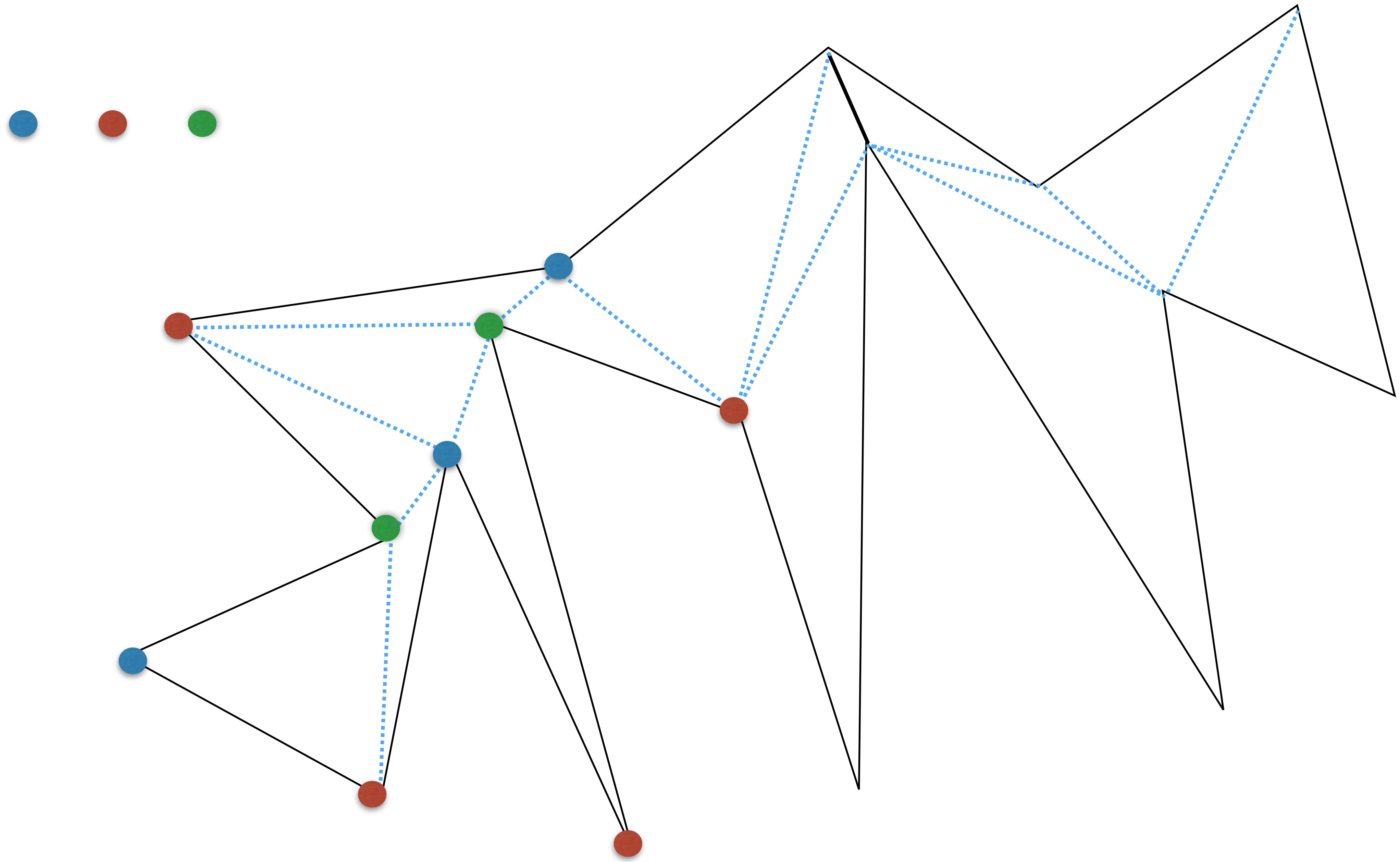

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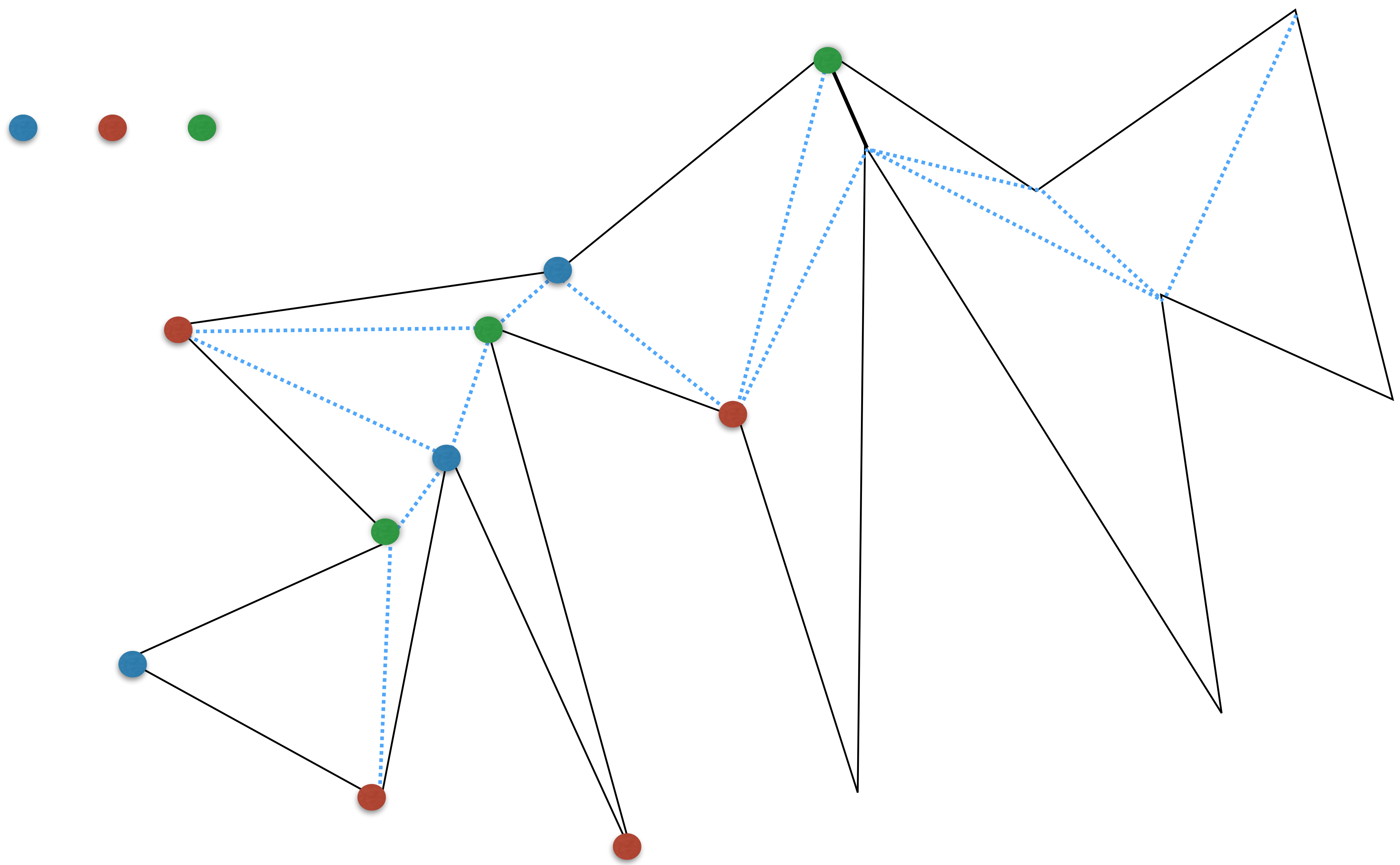

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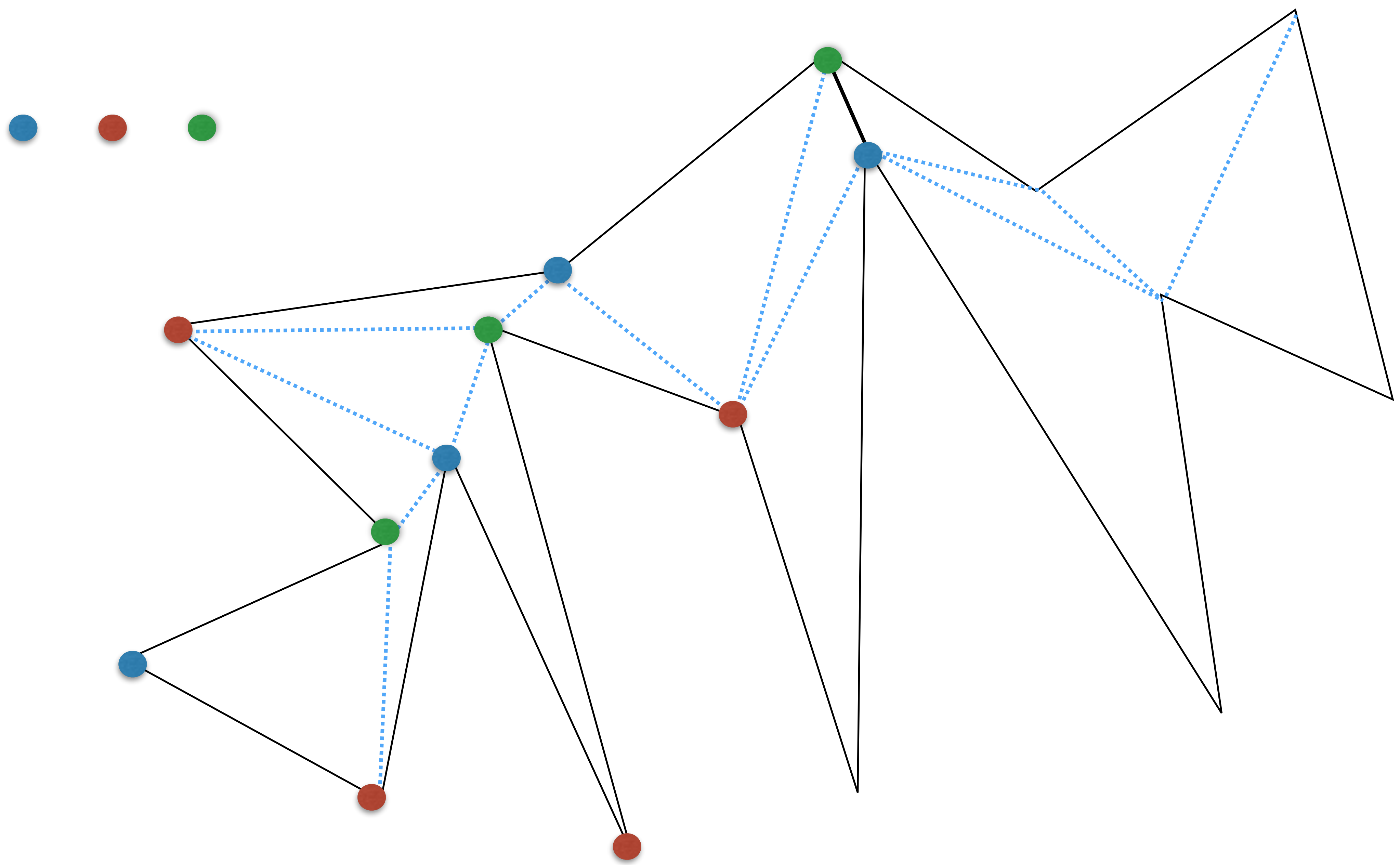

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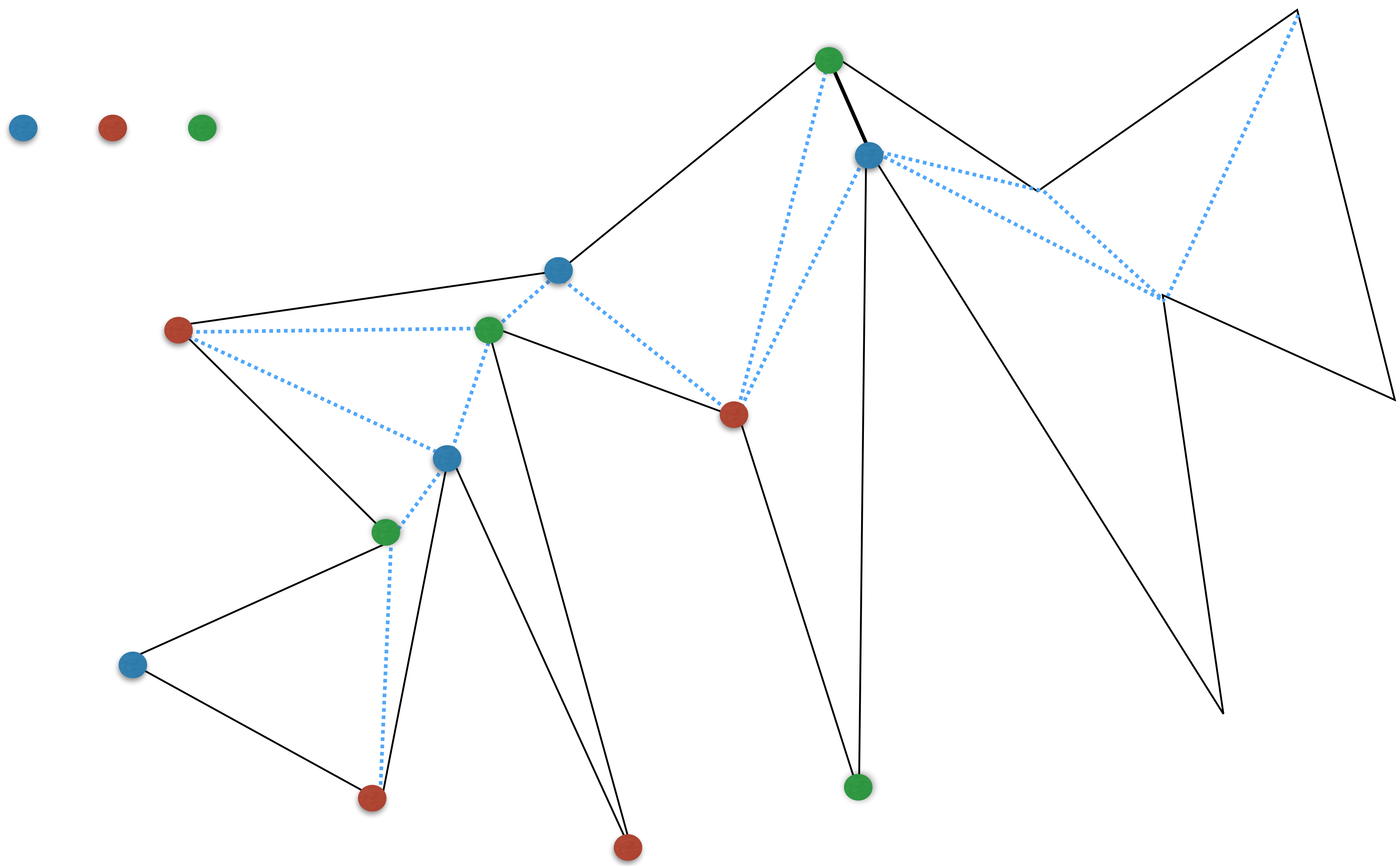

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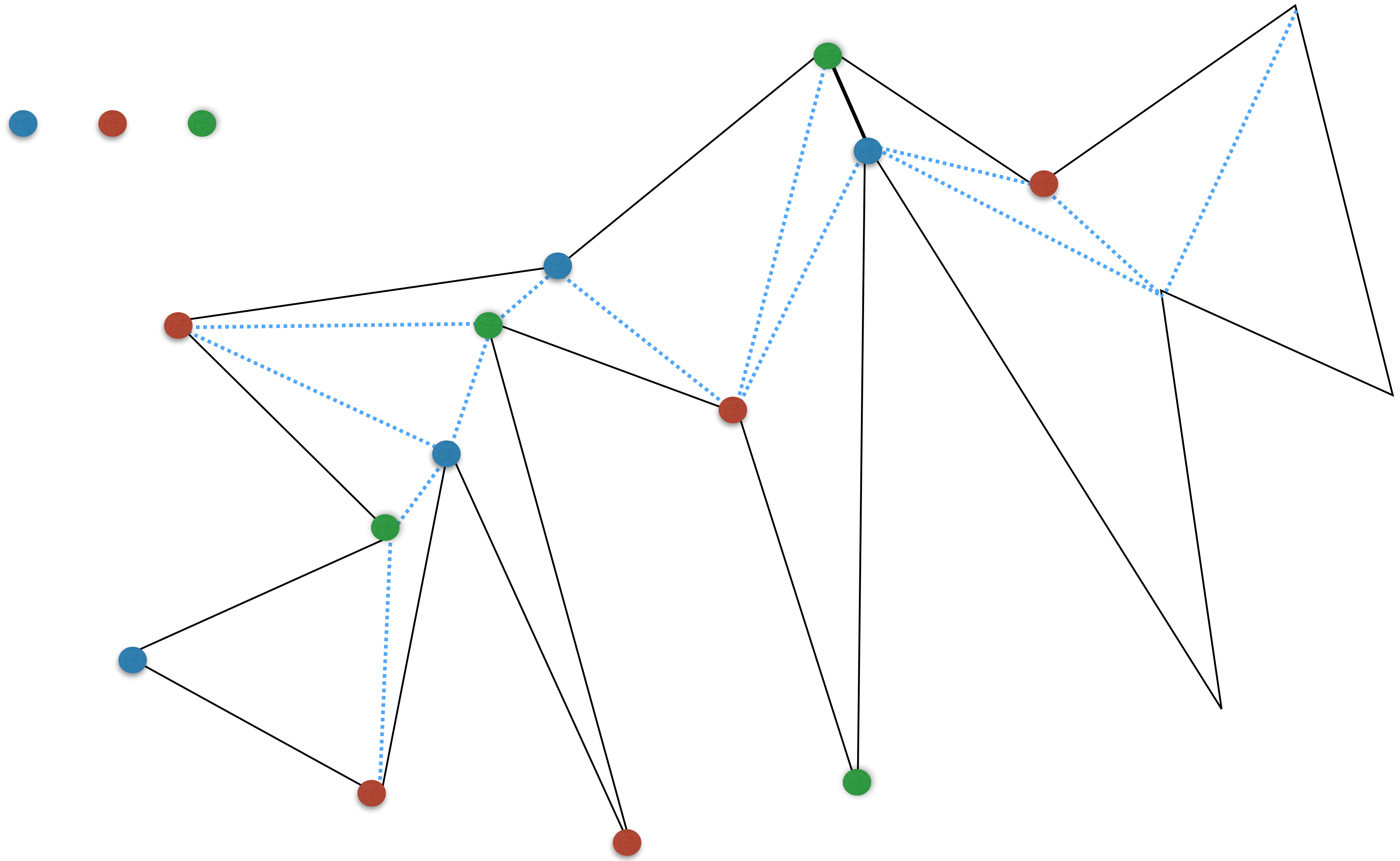

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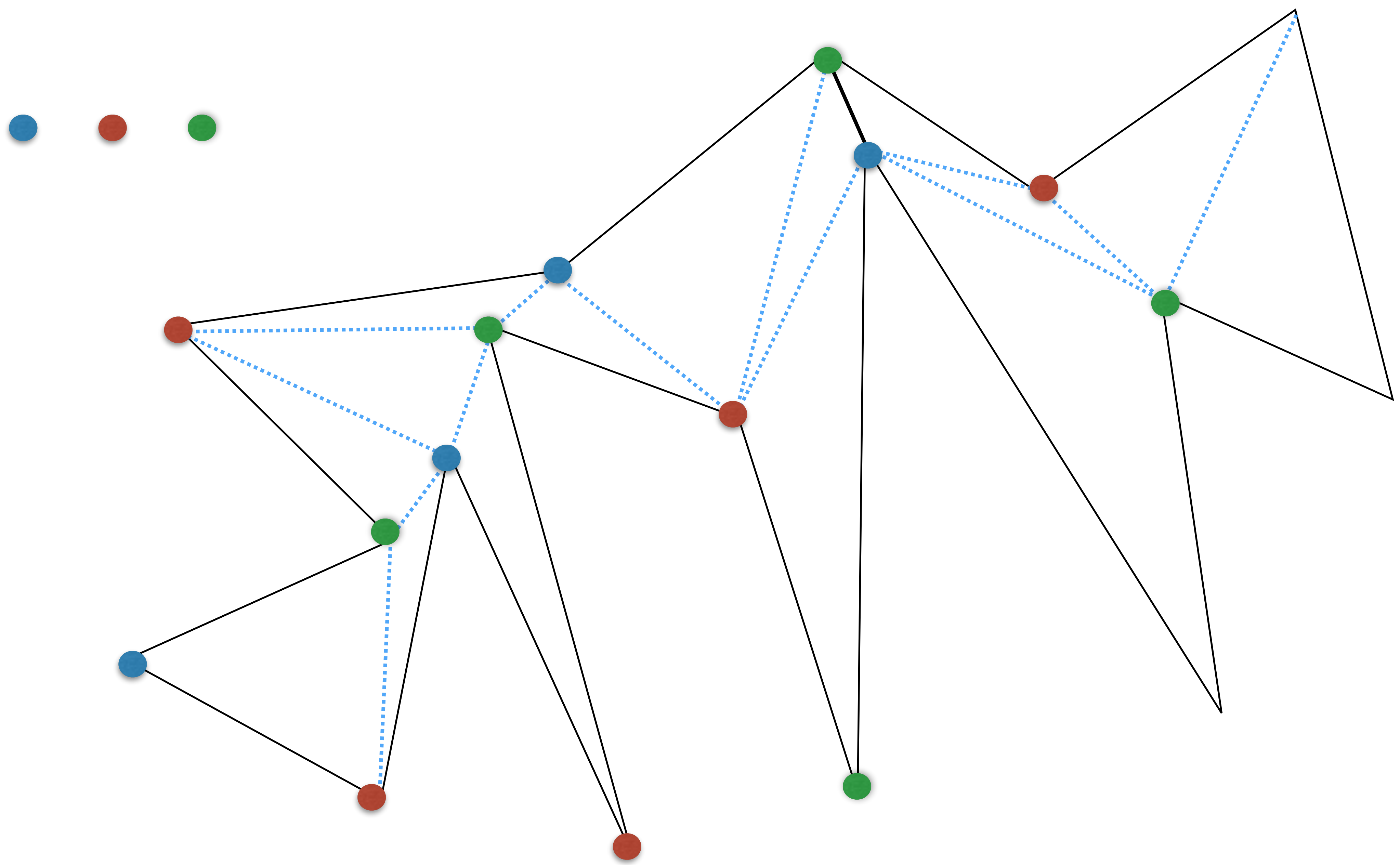

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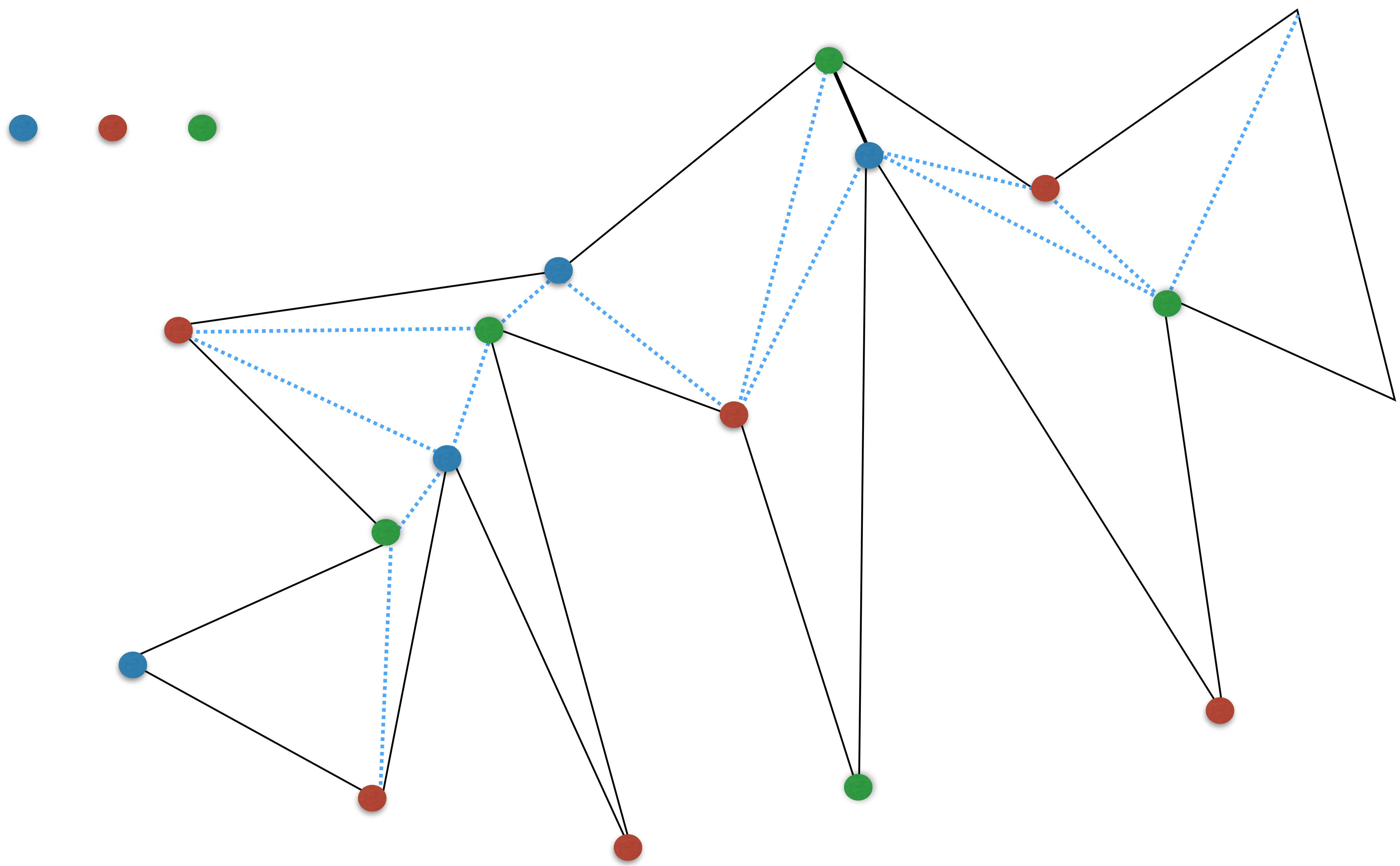

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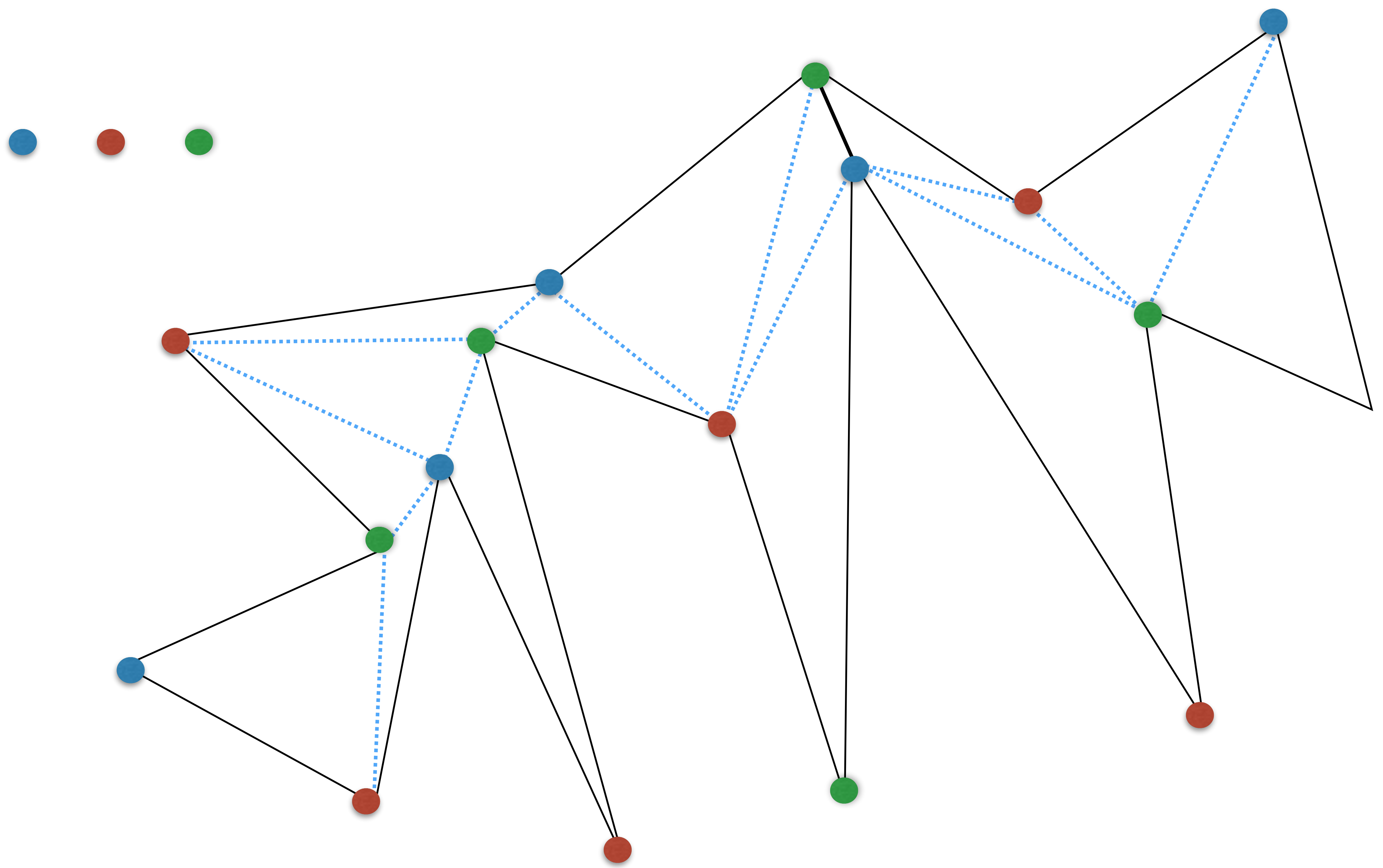

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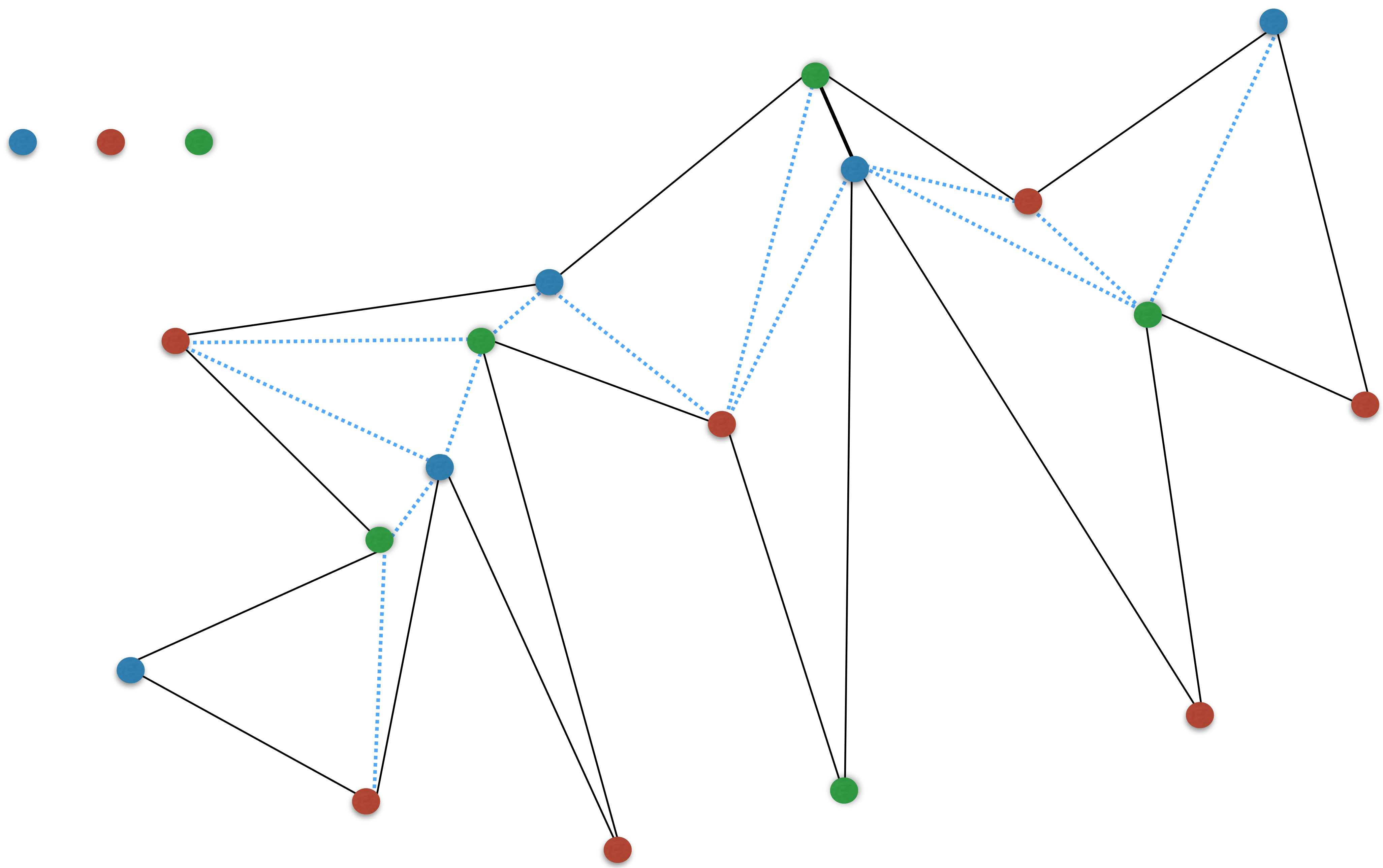

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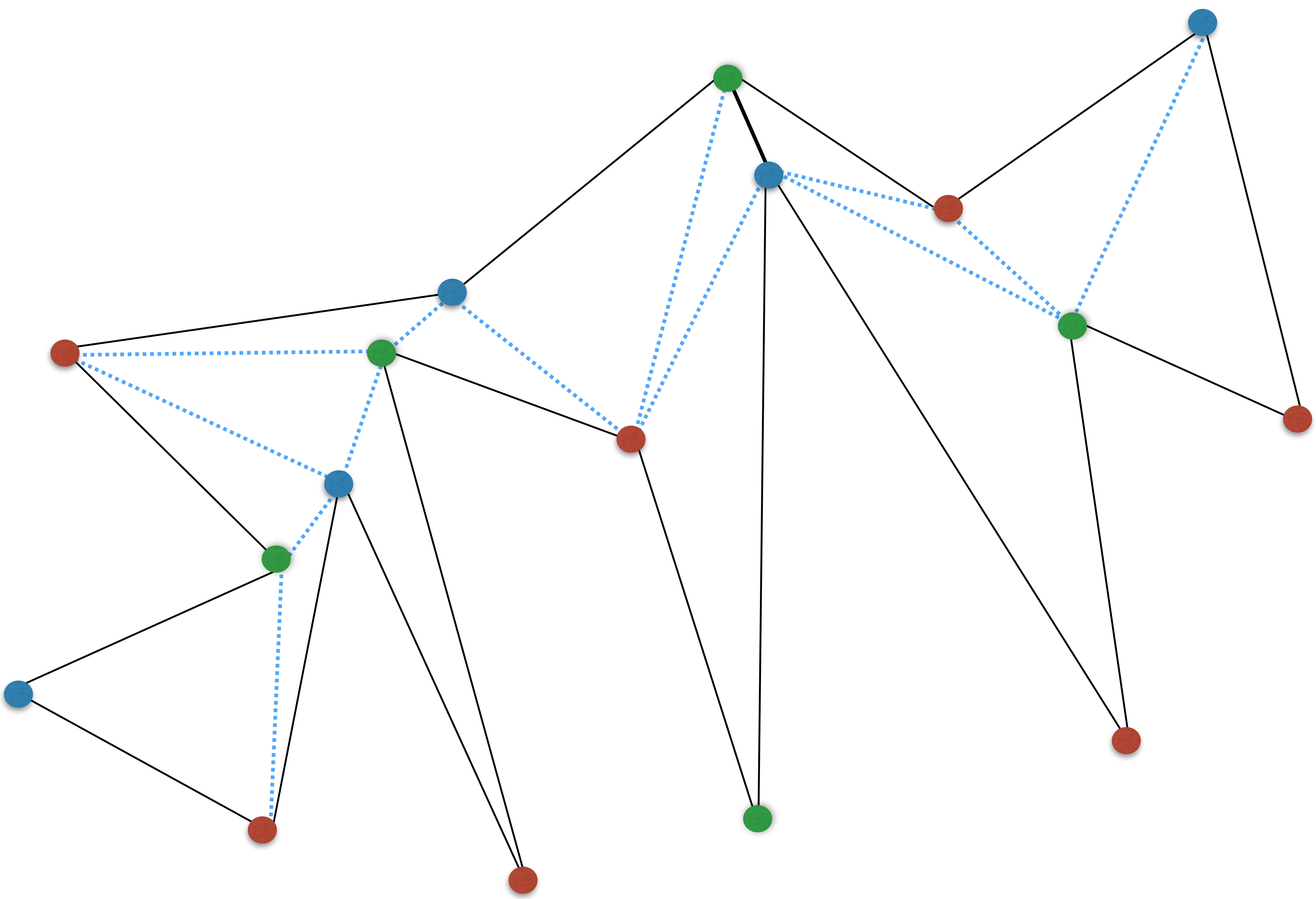

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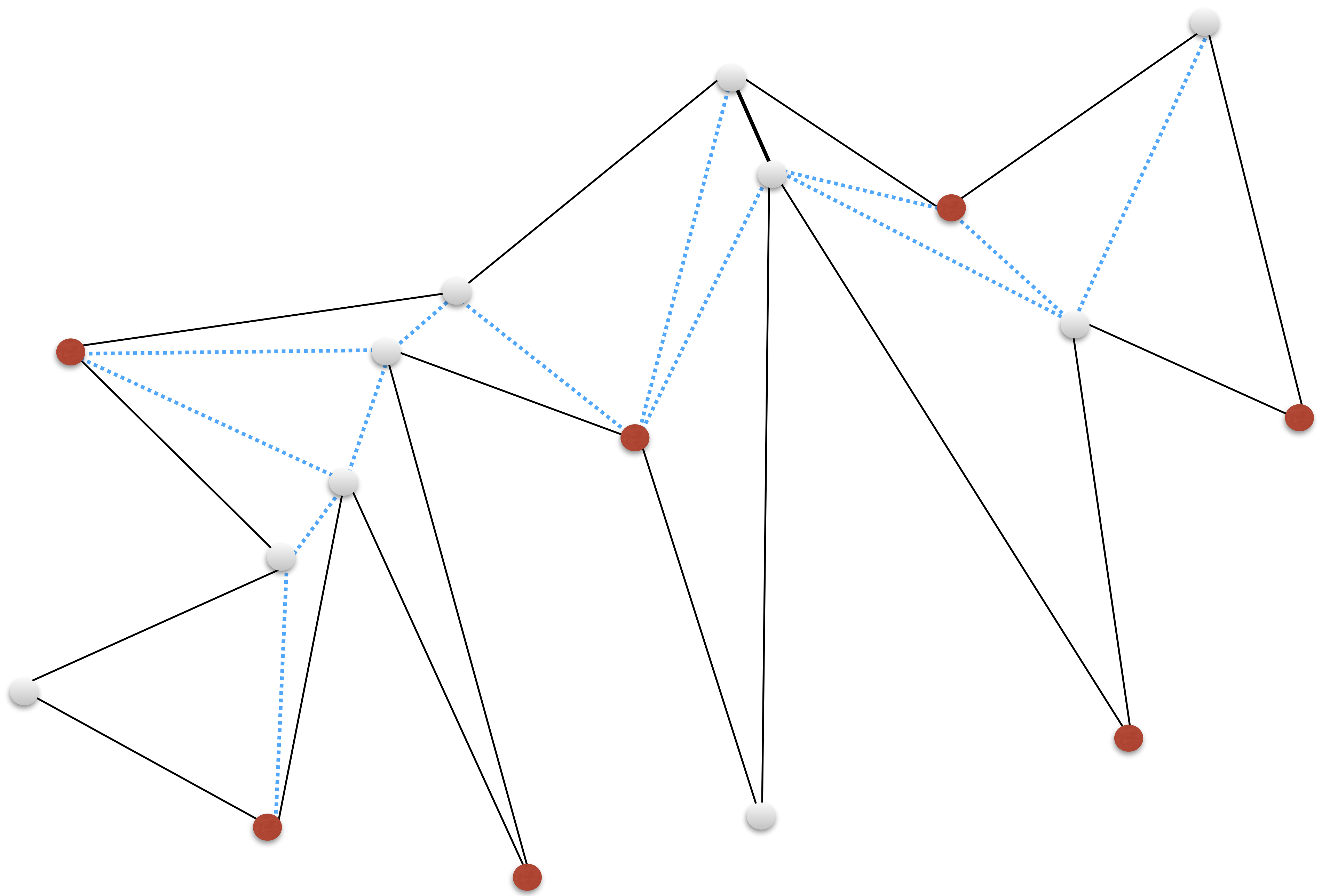

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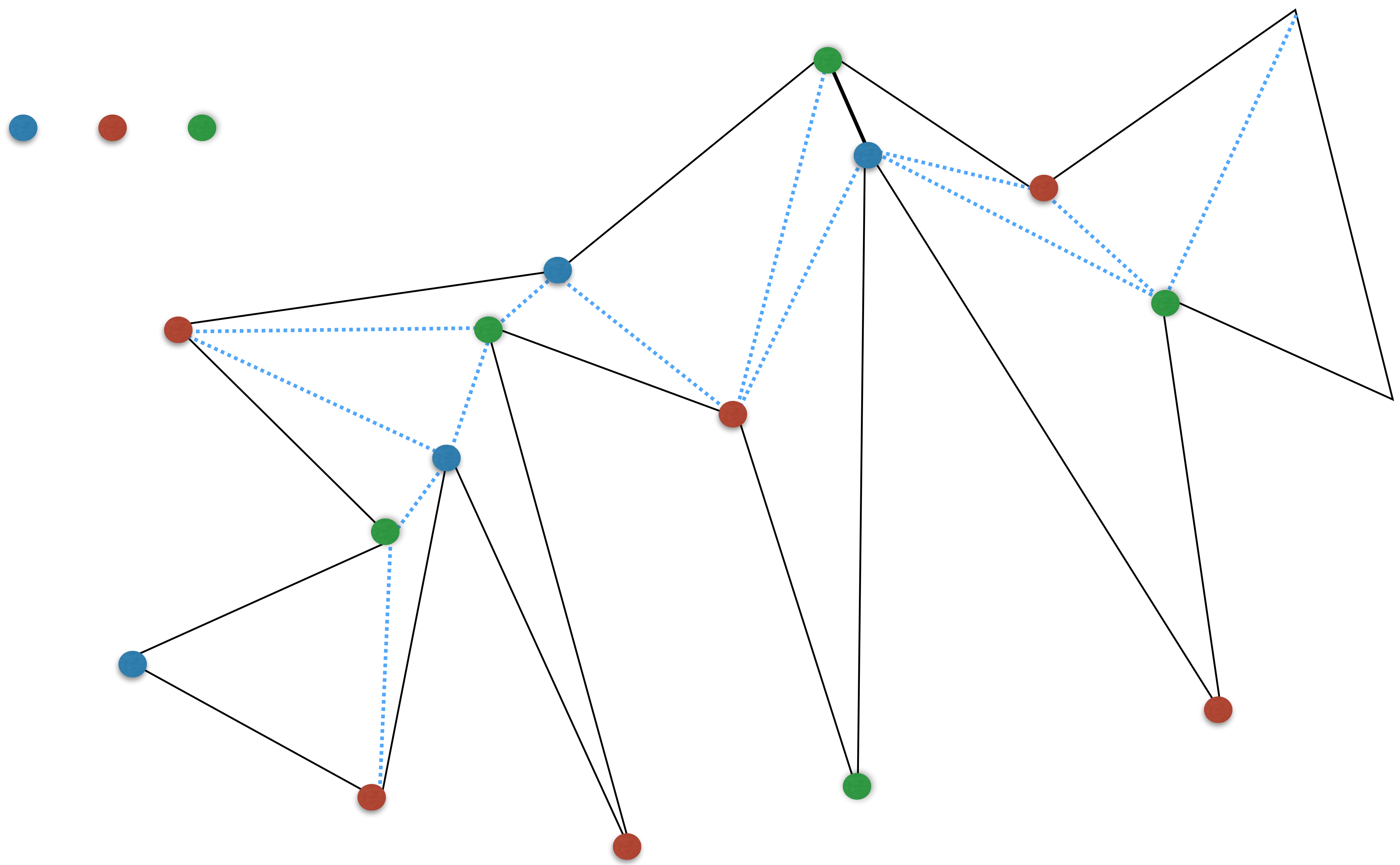
- Placing guards at vertices of one color covers P.
- Pick least frequent color! At most n/3 vertices of that color.



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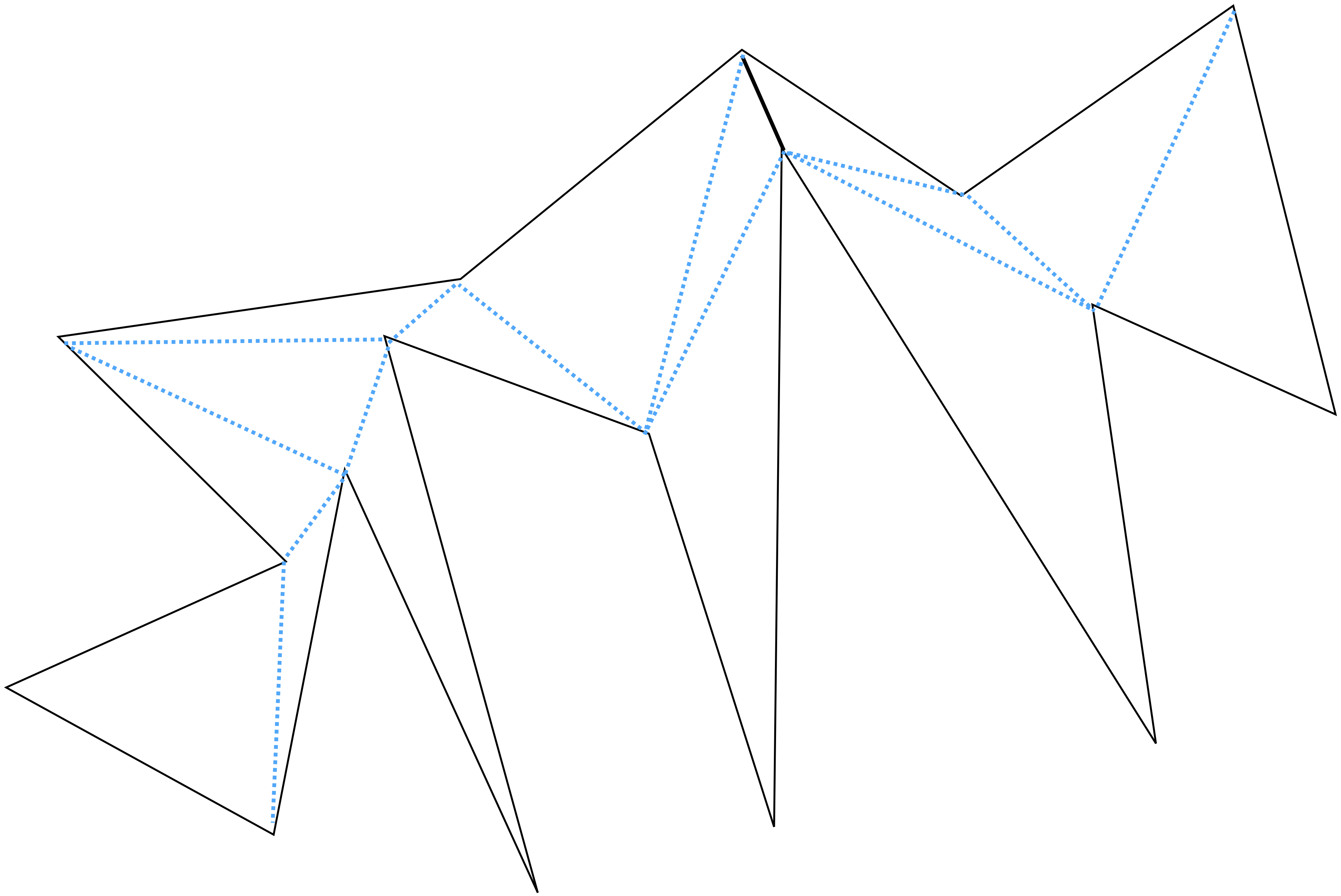
# The proofs

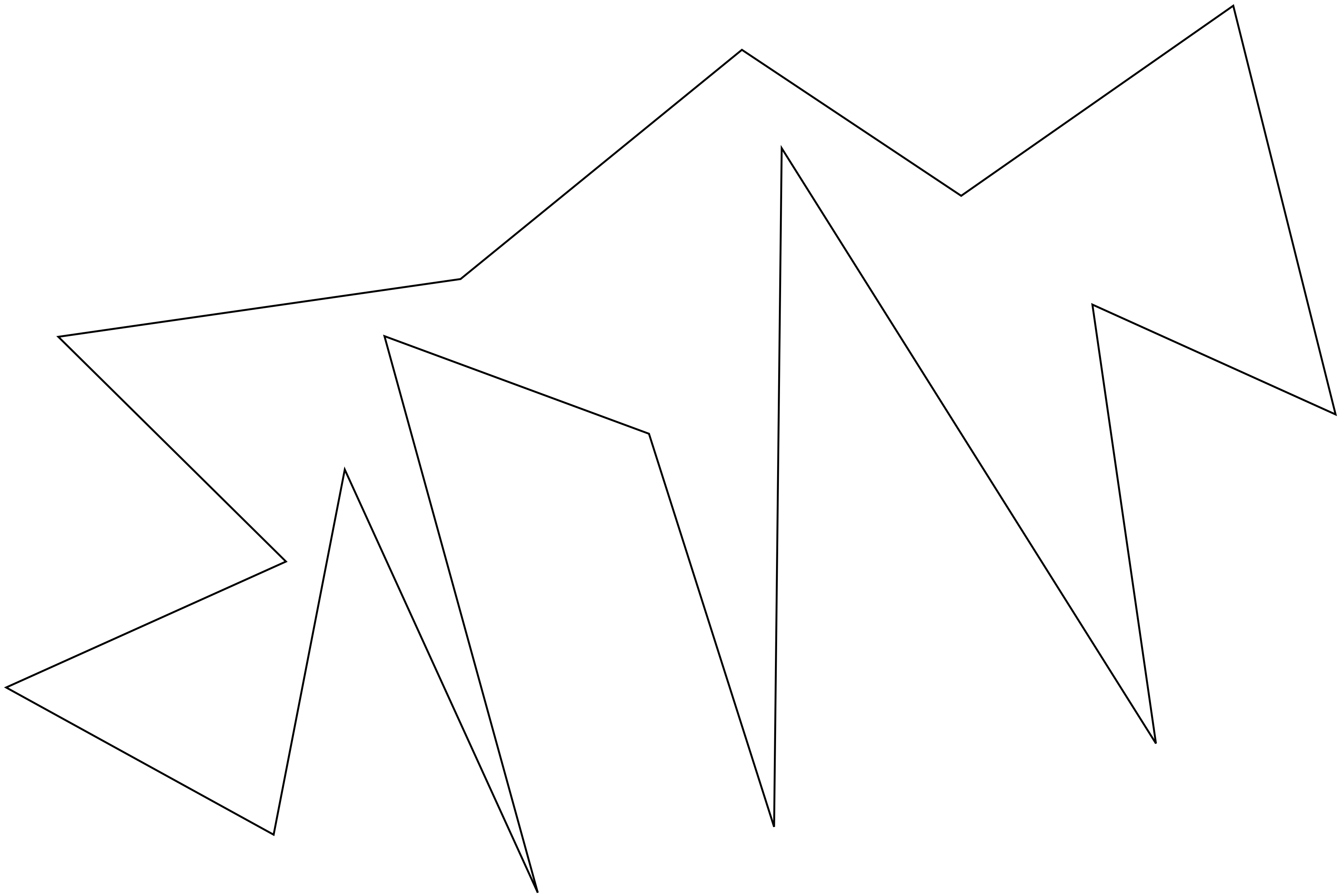
We want to prove that:

Theorem: Any polygon can be triangulated.



# Polygon triangulation

Given a simple polygon P, a **diagonal** is a segment between 2 non-adjacent vertices that lies entirely within the interior of the polygon.



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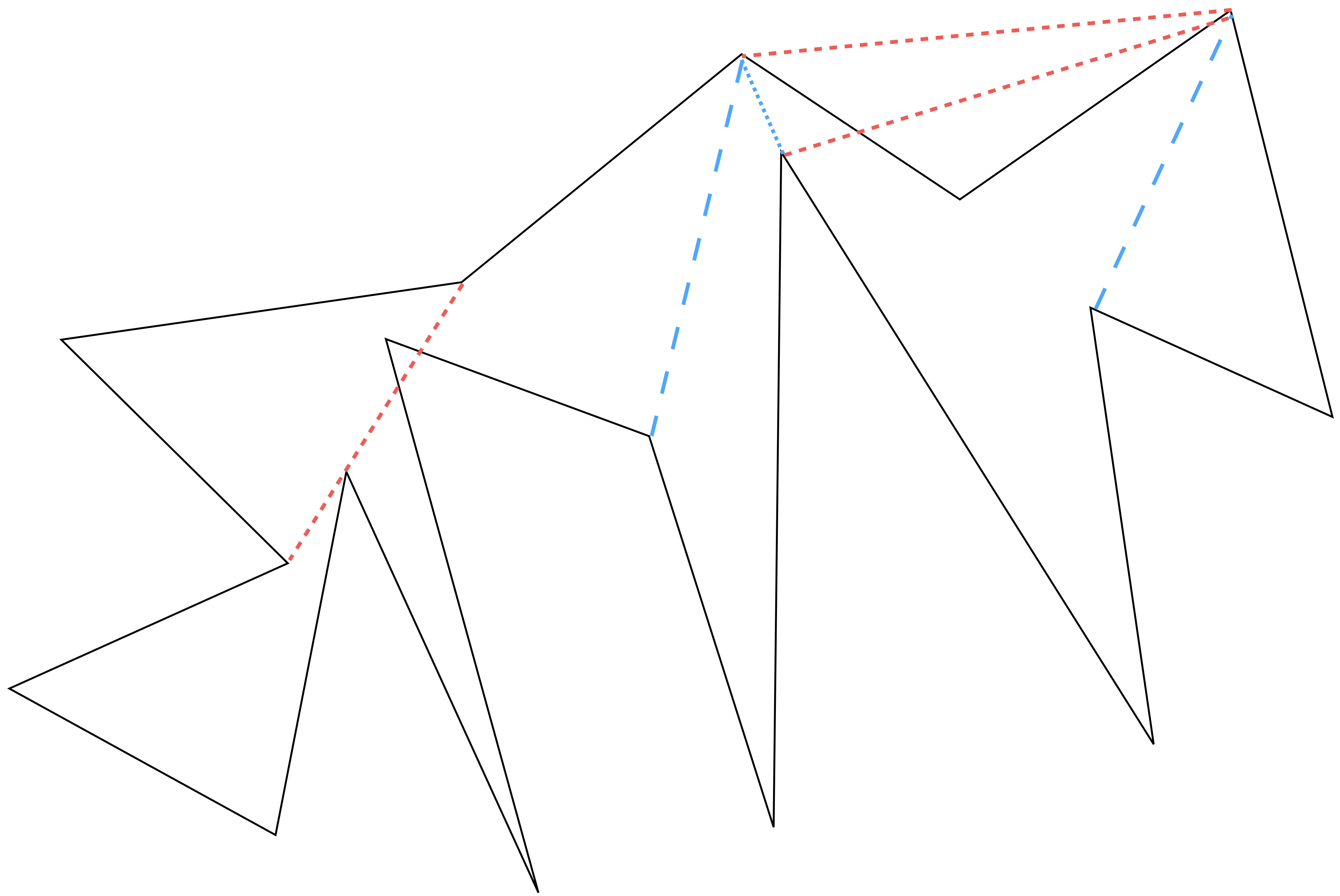

# Polygon triangulation

Theorem: Any simple polygon has at least one convex vertex.

Proof:

# Polygon triangulation

Theorem: Any simple polygon with n>3 vertices contains (at least) a diagonal.

Proof:

# Fisk's proof of sufficiency

Theorem: Any polygon can be triangulated

Proof:

# Fisk’s proof of sufficiency

Theorem: Any triangulation can be 3-colored.

Proof: