Convex hulls in 2D

The problem: Given a set P of n points in the plane, find their convex hull.

Properties of the convex hull

- A point is on the CH if and only of it is *extreme* (a point p is extreme if there exists a line l through it such that all other points are on or on one side of l).
- An edge is on the CH if and only of it is *extreme* (a line l is extreme if all points in P are on or on one side of it).
- A point p is **not** on the CH if and only if p is contained in the interior of a triangle formed by three other points of P.
- The points with minimum/maximum x-coordinate are on the CH.
- The points with minimum/maximum y-coordinate are on the CH.
- Walking counter-clockwise on the boundary of the CH you make only left turns.
- Consider a point p inside the CH. The points on the boundary of the CH are encountered in sorted radial order wrt p.

Algorithms

We discussed the following algorithms:

Brute force

Idea: Find all extreme edges

Algorithm BruteForce (input: points P)

- for all distinct pairs of points (p_i, p_j):
  - if edge (p_i, p_j) is extreme, output it as CH edge

Questions:

- How do you check if an edge is extreme, and how fast?
- What is the overall running time of Algorithm BruteForce?

Gift wrapping

Idea: start from a point p guaranteed to be on the CH and find the edge pq of the CH starting at p; repeat from q.

> Algorithm GiftWrapping (input: points P)
>
> - Let p_0 be the point with smallest x-coordinate (if more than one, pick right-most)
> - $p = p_0$
> - repeat
>
>   for each point q, $q! = p$:
>
>   - compute counter-clockwise-angle of q wrt p
>
>   let p' be the point with smallest such angle
>
>   //claim: edge (p, p') is on the CH because...
>
>   output (p, p') as CH edge
>
> $p = p'$
> - until $p == p_0$

Questions:

1. Simulate GiftWrapping on a set of points and check that it works in degenerate cases.
2. What is the running time of Algorithm GiftWrapping? Express the running time as function of k, where k is the output size (in the case the size of the CH). This is called an *output-sensitive* bound and GiftWrapping's runnung time is output-sensitive.
3. How big/small can k be for a set of n points? Show examples that trigger best/worst case for GiftWrapping.
4. Discuss when GiftWrapping is a good choice.

QuickHull

Idea: Similar to Quicksort. Partition, then recurse.

Algorithm QuickHull (input: points P)

- Find left-most point a and right-most point b
- Partition P into P_1 (points left of ab) and P_2 (points right of ab)
- return QuickHull(a, b, P_1) + QuickHull(b, a, P_2)

QuickHull(a, b, P)
//invariant: P is a set of points all left of ab

- if P is empty: return emptyset
- for each point $p \in P$: compute its distance to ab
- let c be the point with max distance
- let P_1 = points to the left of ac
- let P_2 = points to the left of cb
- return QuickHull(a, c, P_1) + c + QuickHull(c, b, P_2)

Questions:

- Simulate QuickHull and check that it works in degenerate cases
- Write a recurrence for its running time.
- What is the best/worst case running time of QuickHull? Show examples.
- Argue that Quickhull's average complexity is $O(n)$ when points are uniformly distributed.

Graham scan

Idea: start from a point p interior to the hull. Order all points by their ccw angle wrt p. Traverse and maintain the CH of all traversed points.

Algorithm GrahamScan (input: points P)

- Find interior point p_0 (instead of an interior point, can pick the lowest point)
- Sort all other points ccw around p_0 and call them $p_1, p_2, ...p_{n-1}$ in this order.
- Initialize stack $S = (p_2, p_1)$
- for i = 3 to n-1 do
  - if p_i is left of (second(S), first(S)): push p_i on S
  - else:
    - repeat: pop S while p_i is right of (second(S), first(S)
    - push p_i on S

Questions:

- Degenerate cases: Simulate the algorithm on some degenerate cases and check that it works (if not, fix it).
- Argue that once the points are sorted, the algorithm takes linear time.

  What is the overall running time of Graham Scan? Is the algorithm output sensitive?