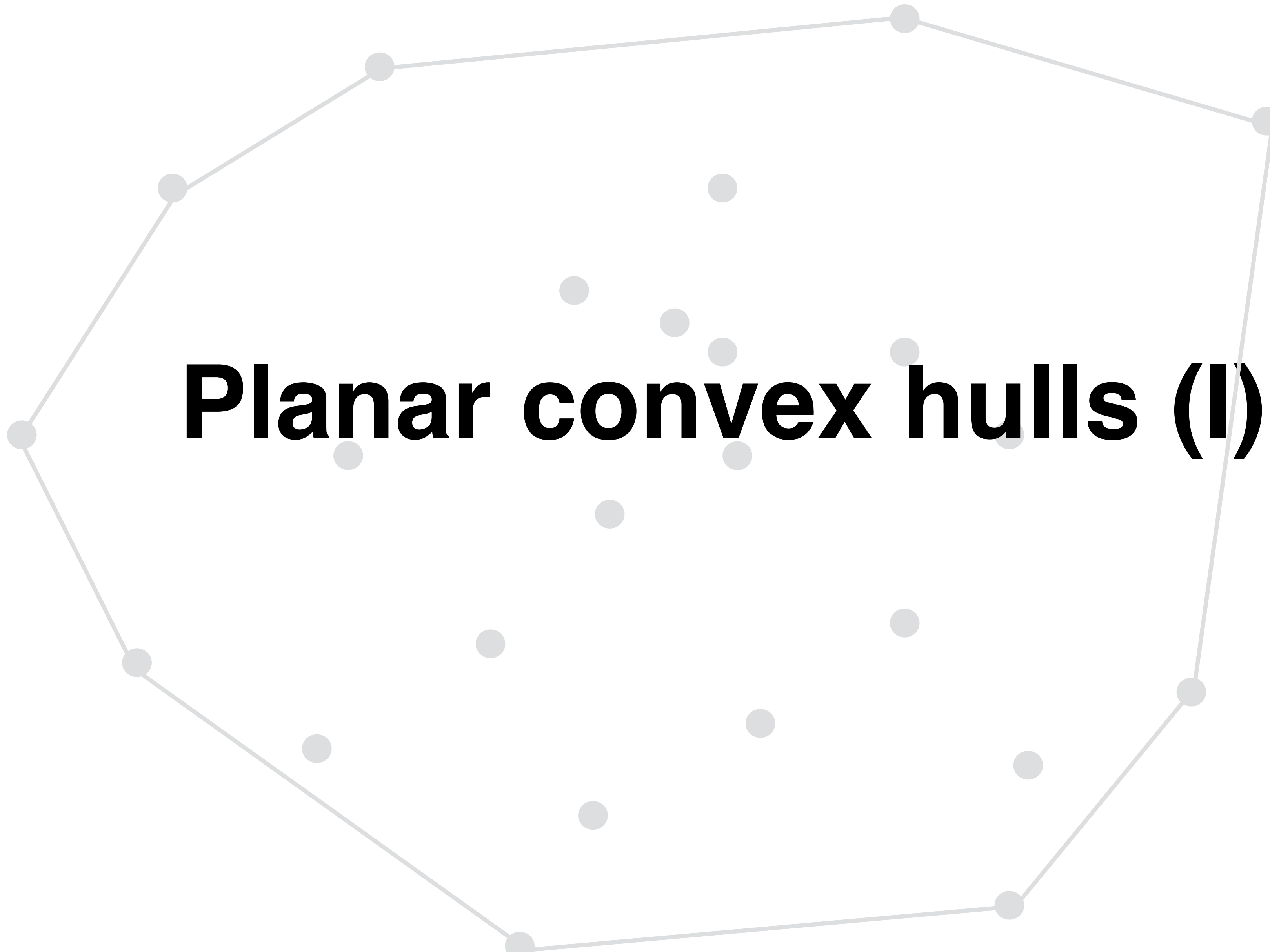
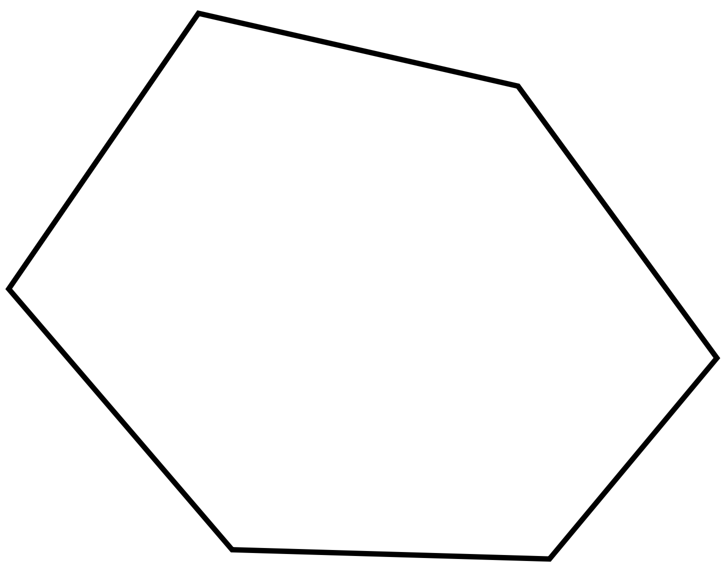
Planar convex hulls (I)



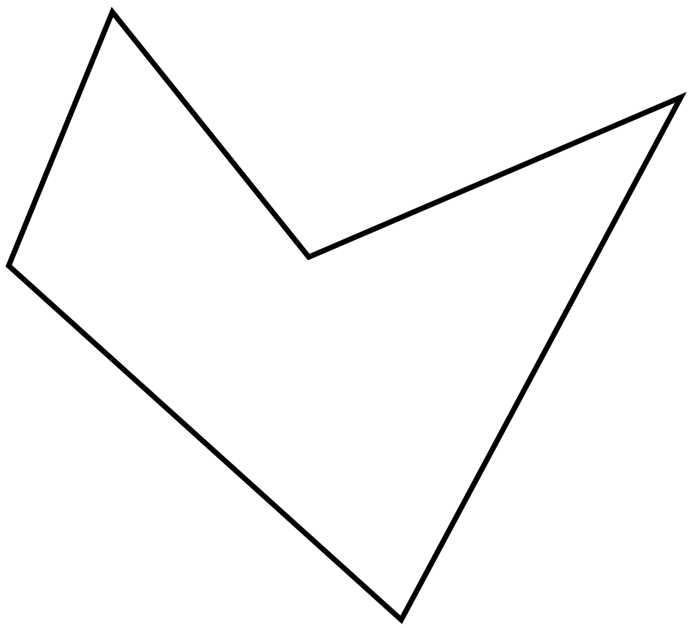
Computational Geometry [csci 3250]
Laura Toma
Bowdoin College

Convexity

A polygon P is **convex** if for any p, q in P, the segment pq lies entirely in P.



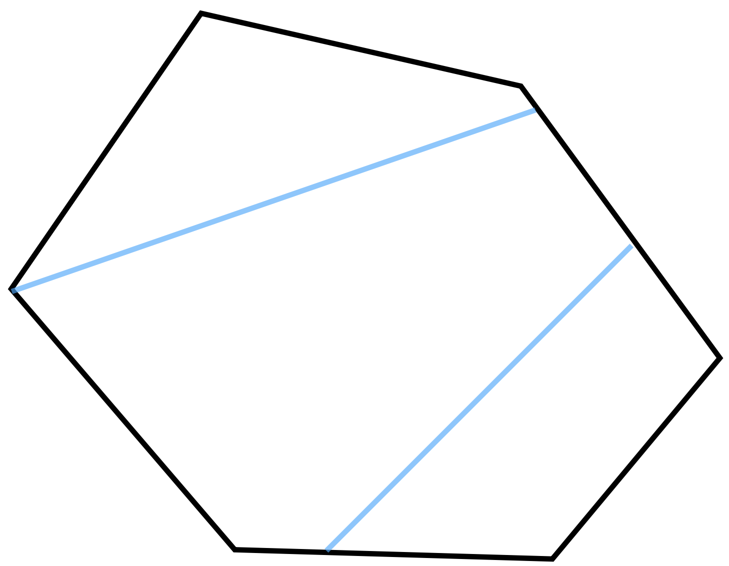
convex



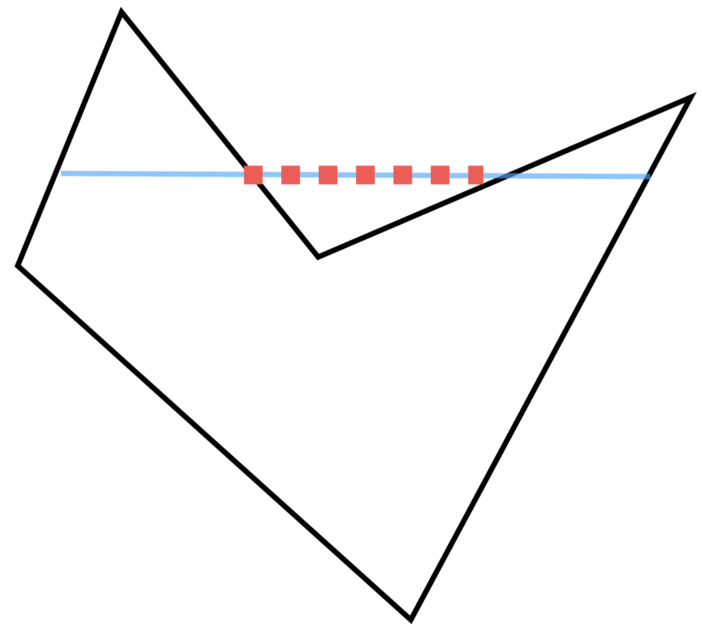
non-convex

Convexity

A polygon P is **convex** if for any p, q in P, the segment pq lies entirely in P.



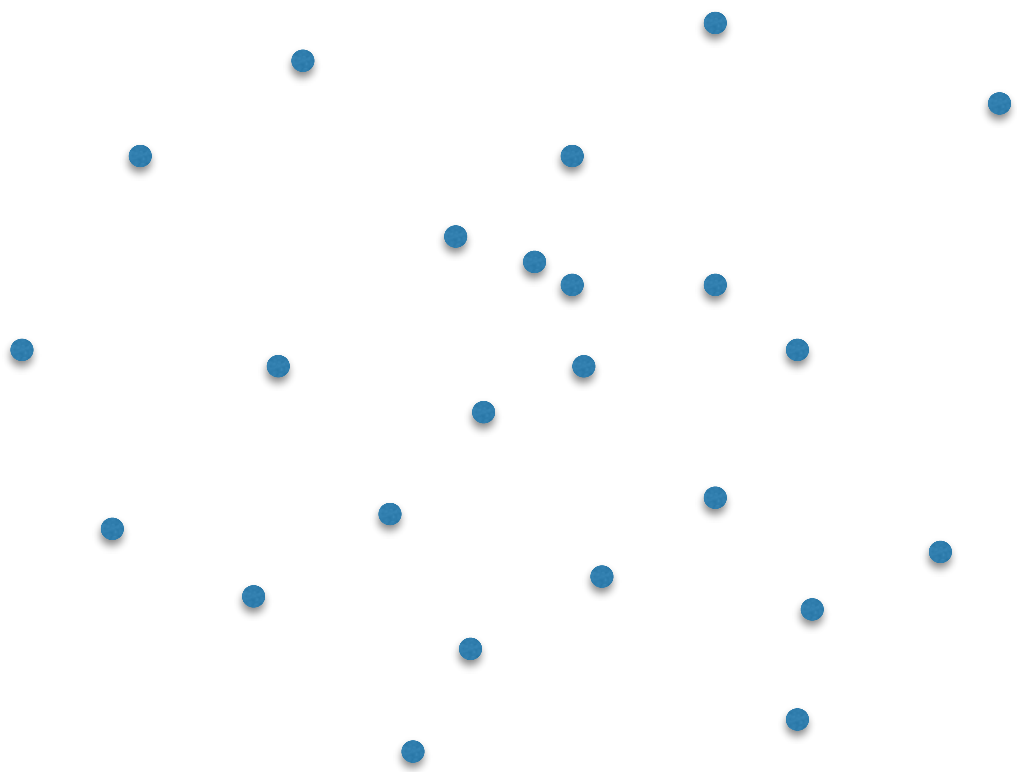
convex



non-convex

Convex Hull

Given a set P of points in 2D, their convex hull is the smallest convex polygon that contains all points of P



Convex Hull

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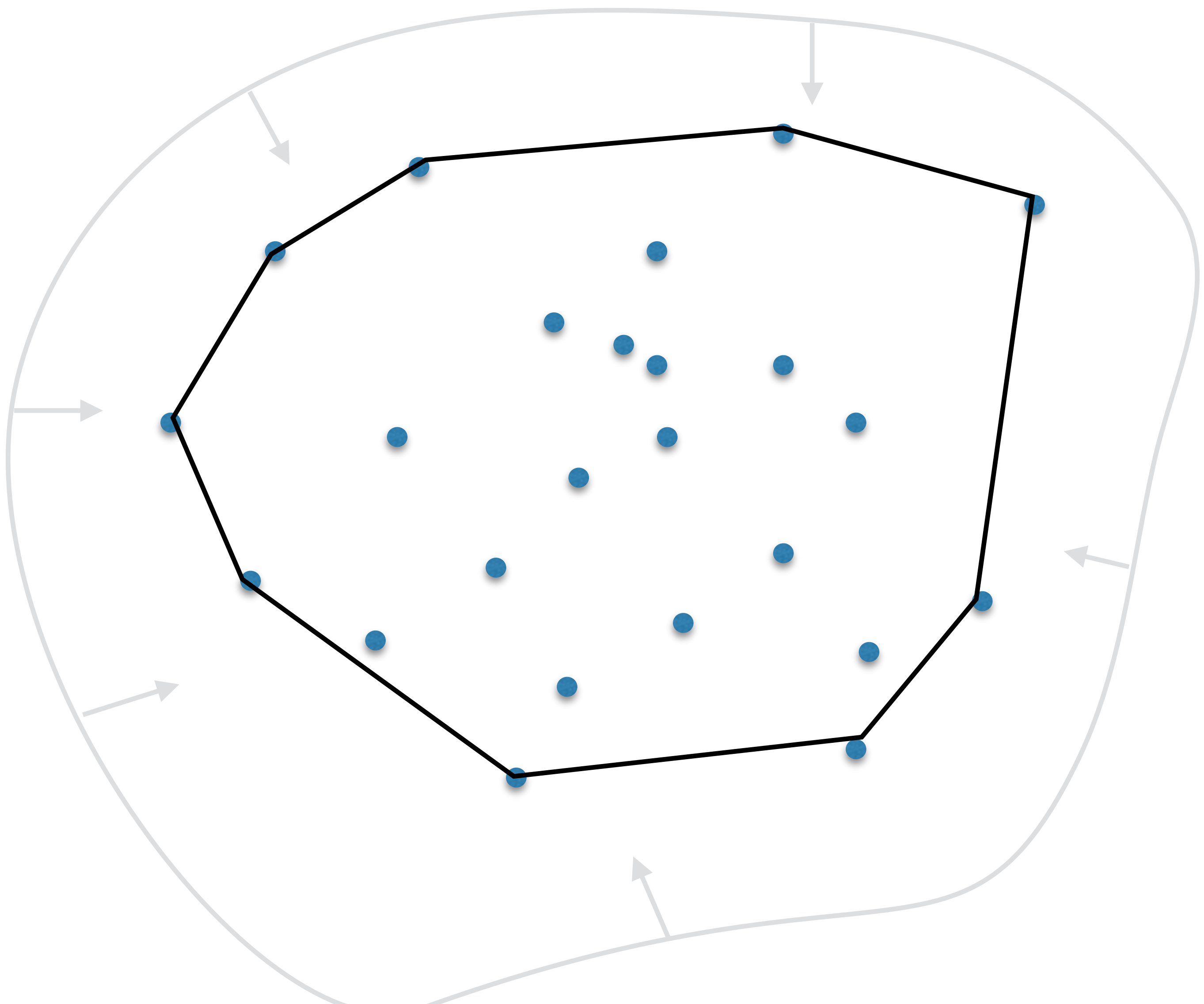
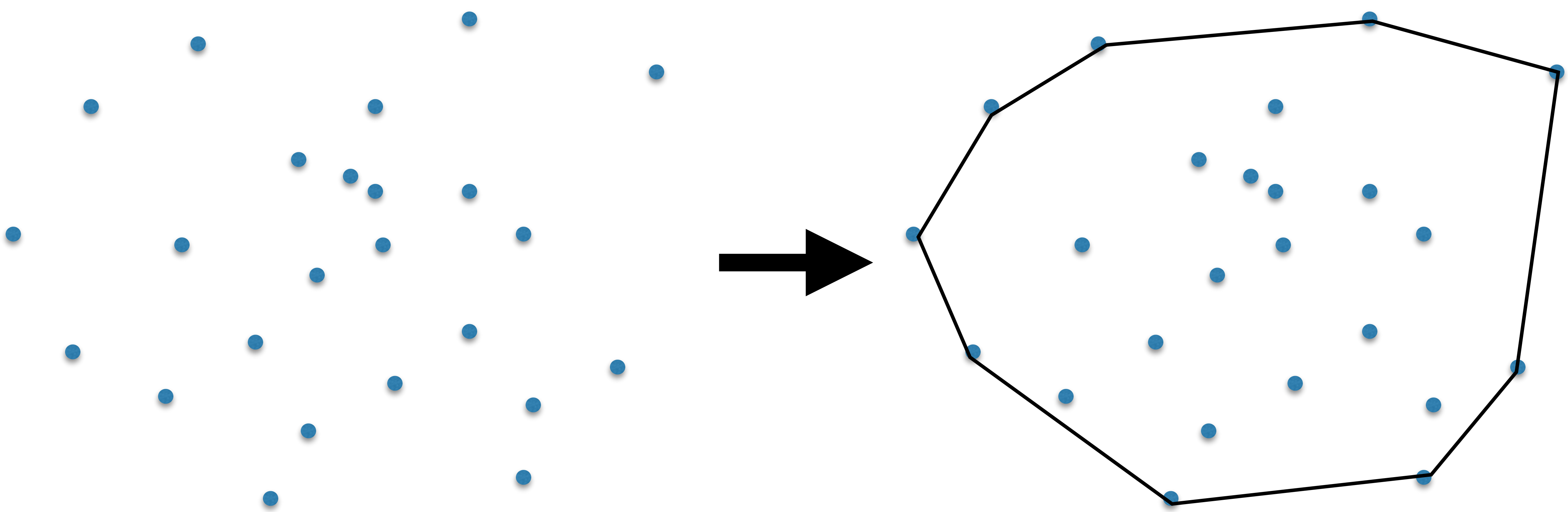


Convex Hull

The problem: Given a set P of points in 2D, describe an algorithm to compute their convex hull



Input:
array P of points (in 2D)

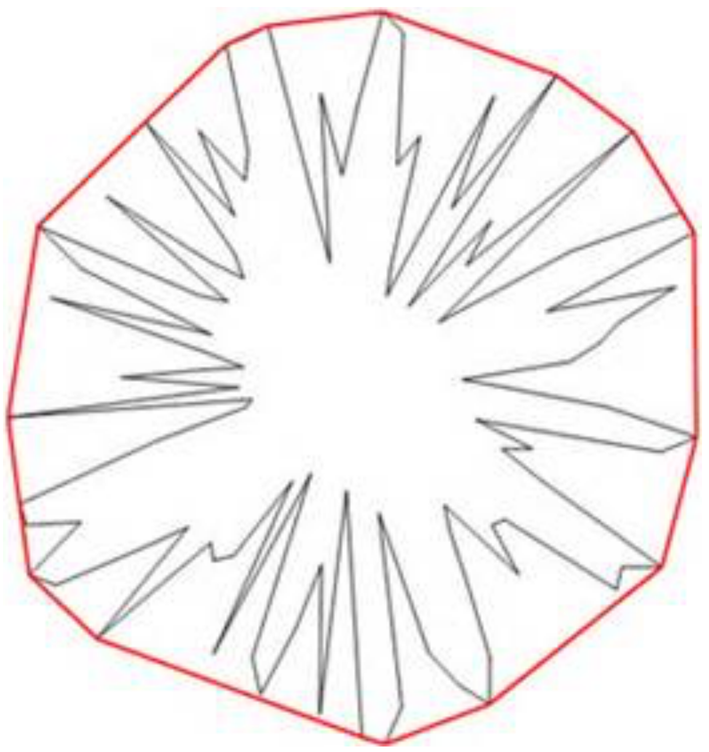
Output:
array/list of points on the CH (in boundary order)

Convex Hull

- One of the first problems studied in CG
- Many solutions
  - simple, elegant, intuitive
  - illustrate techniques for geometrical algorithms
- Lots of applications
  - robotics, path planning, partitioning problems, shape recognition, separation problems, etc

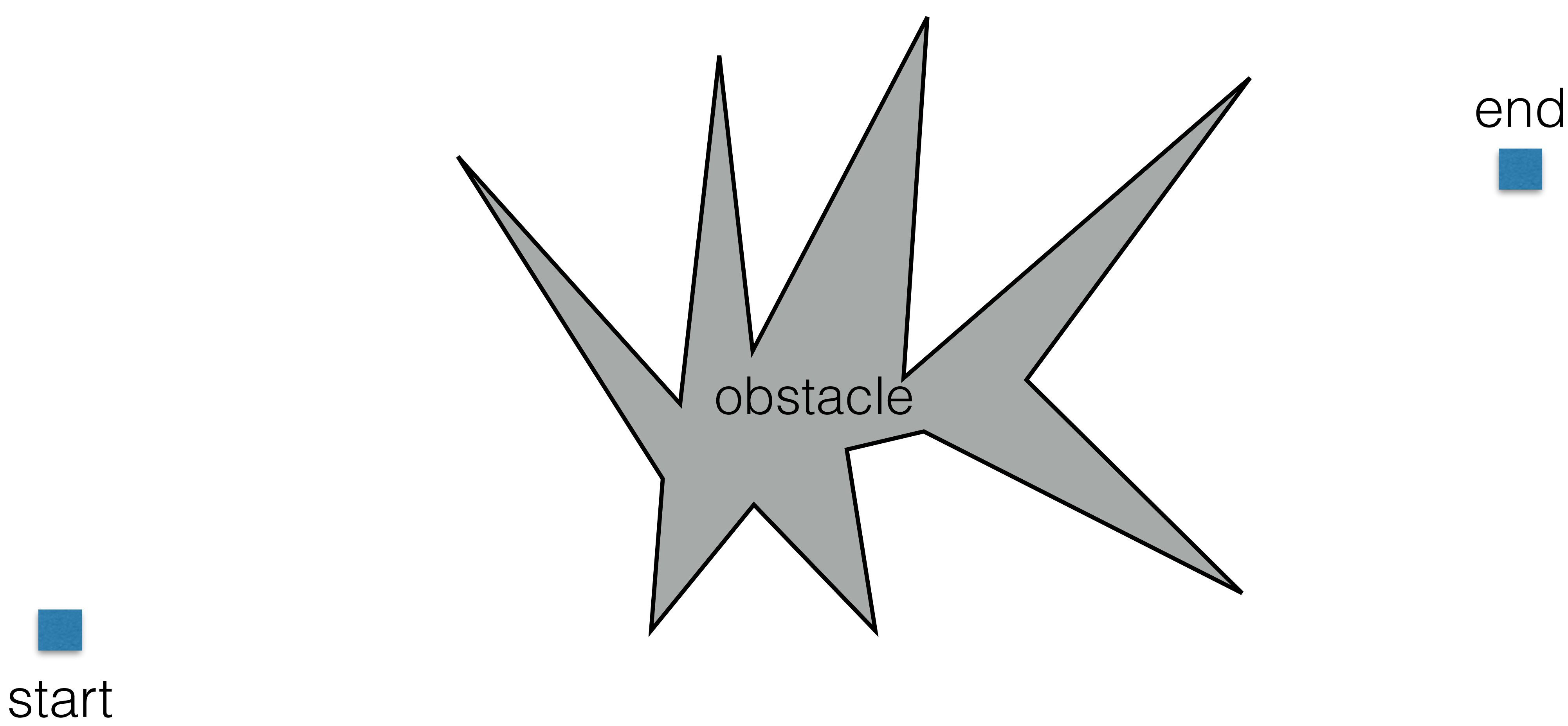
Applications

- Shape analysis, matching, recognition
    - approximate objects by their CH



Applications

- Path planning: find (shortest) collision-free path from start to end



Applications

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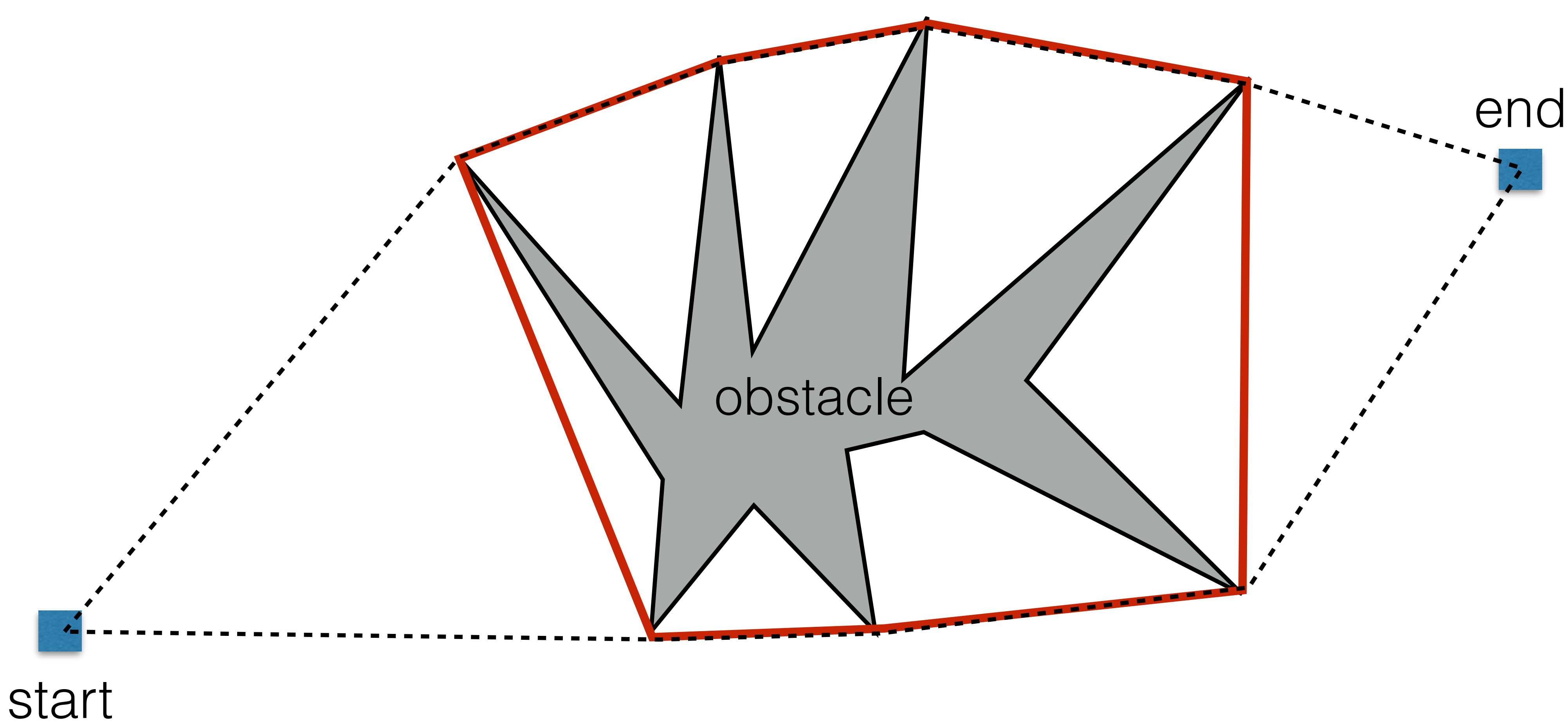
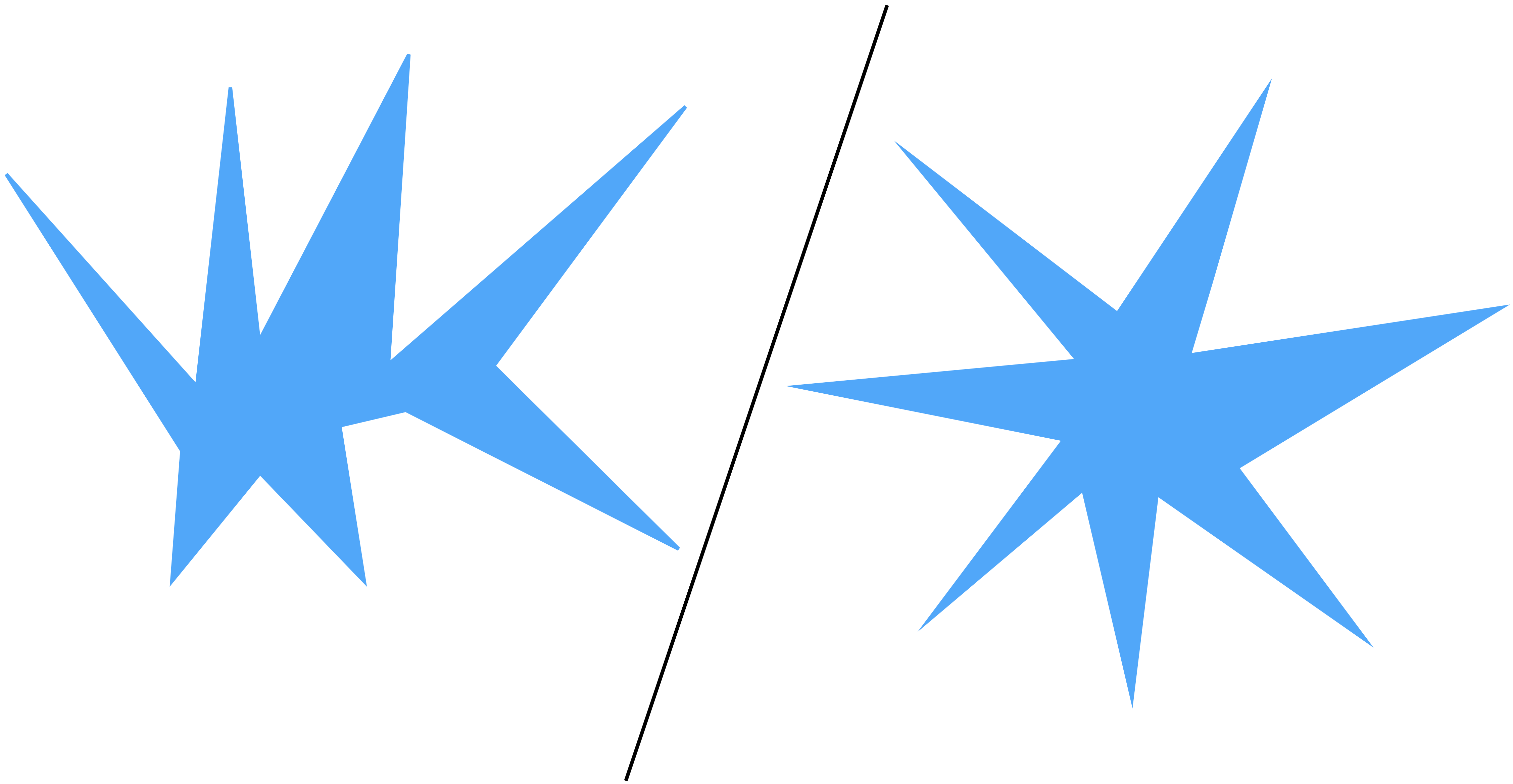


- It can be shown that the shortest path follows CH(obstacle); also, it is the shorter of the upper path and lower path

Applications

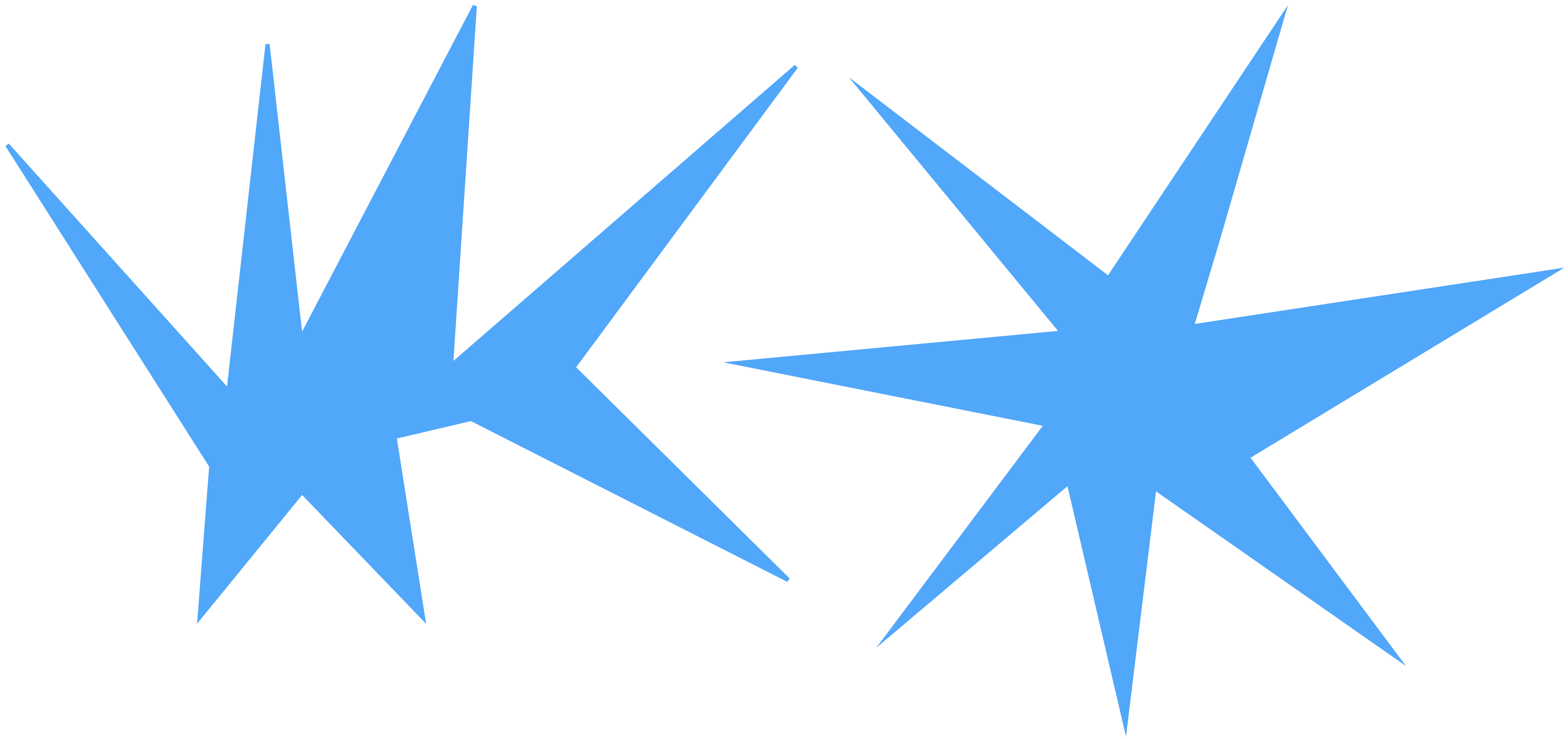
- Partitioning problems
  - does there exist a line separating two objects?



YES

Applications

- Partitioning problems
  - does there exist a line separating two objects?



NO

Applications

- Partitioning problems
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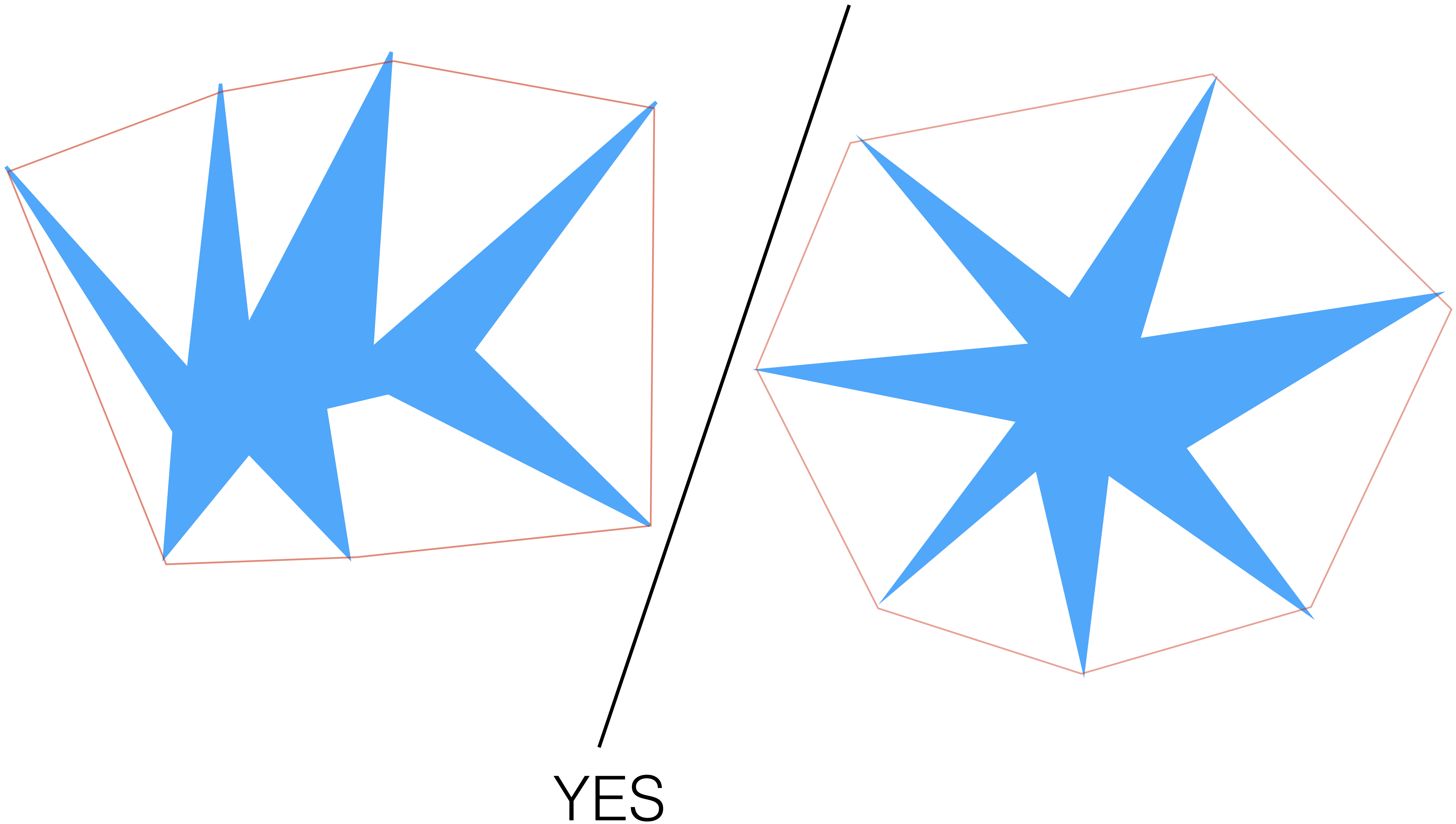

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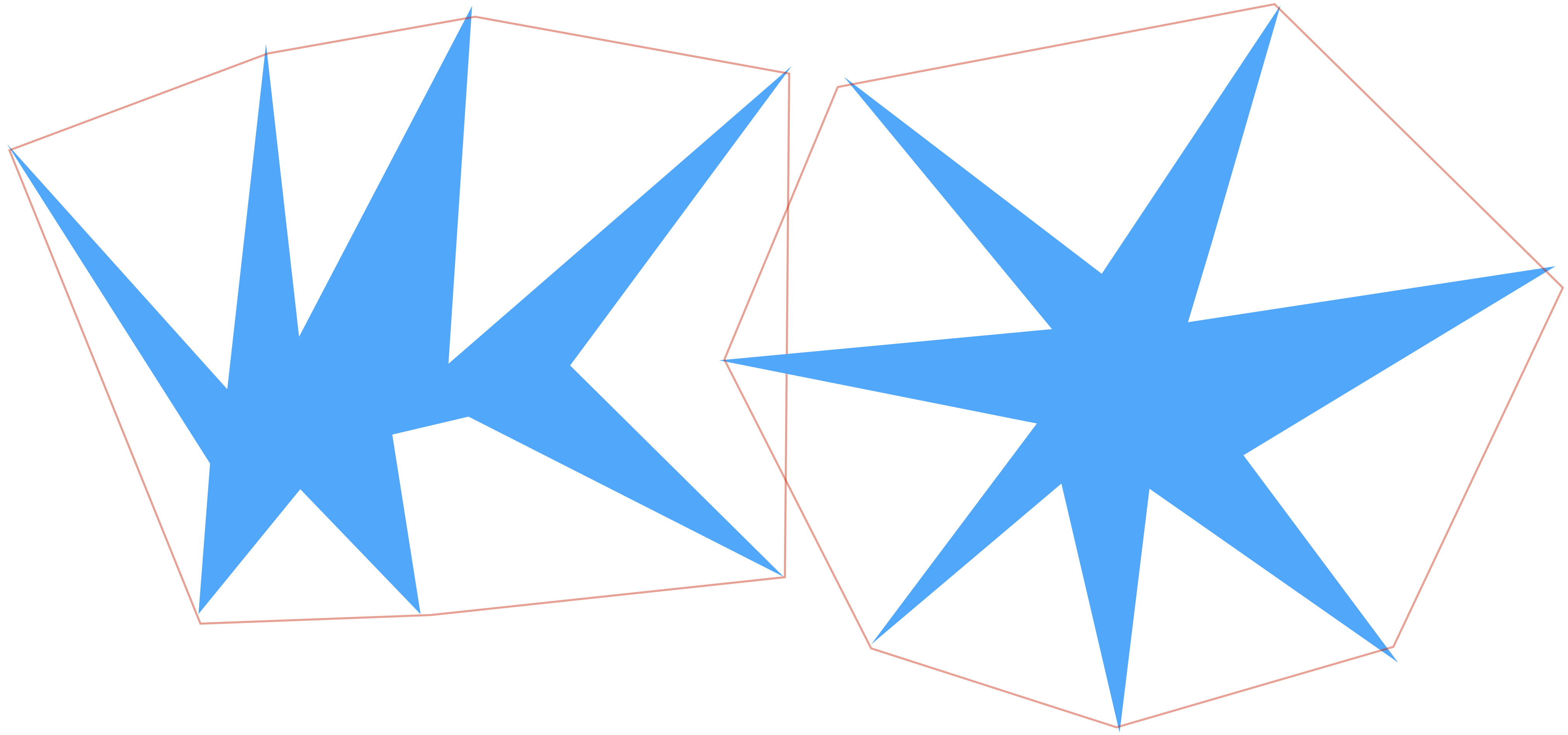
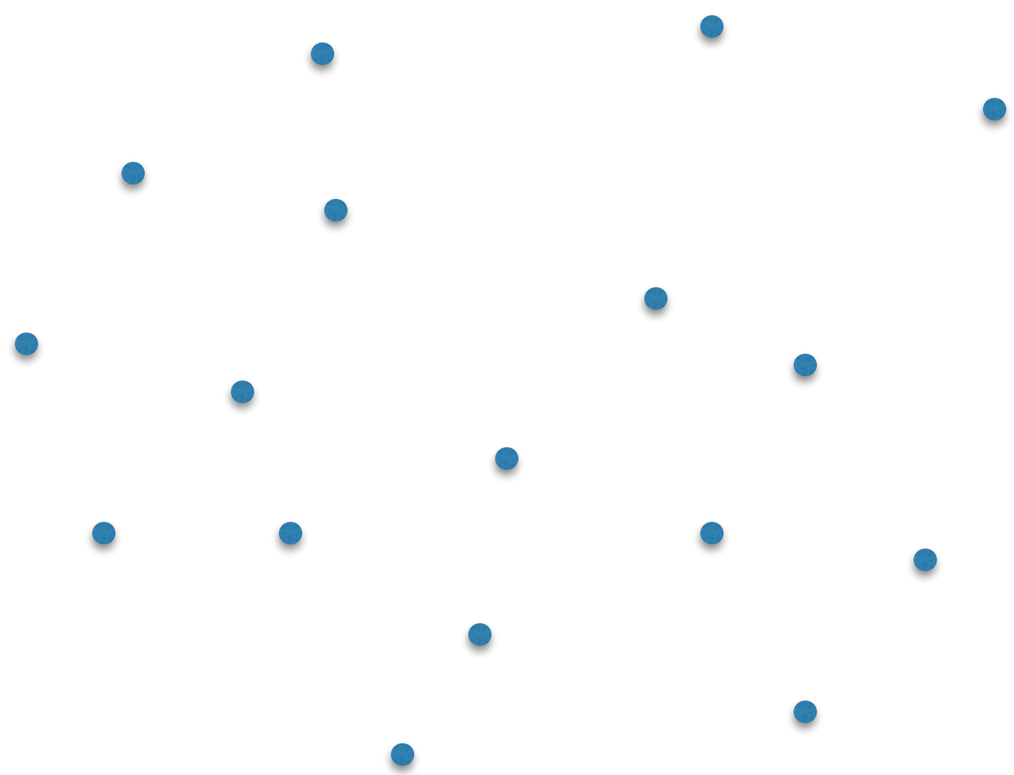


NO

Applications

- Find the two points in P that are farthest away



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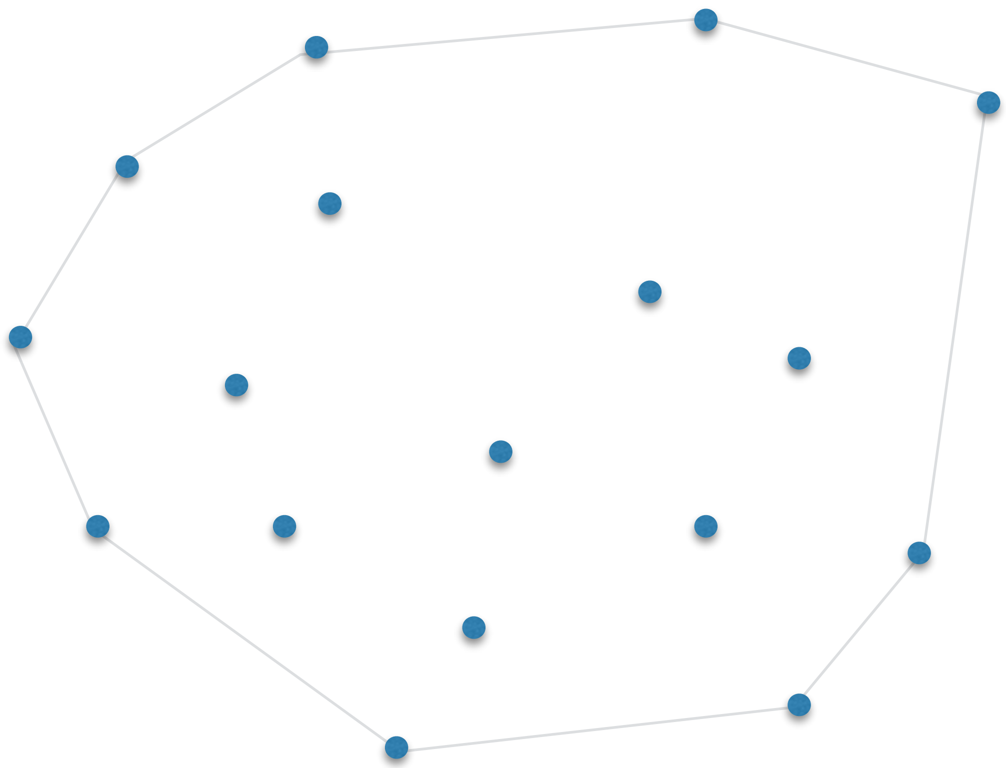


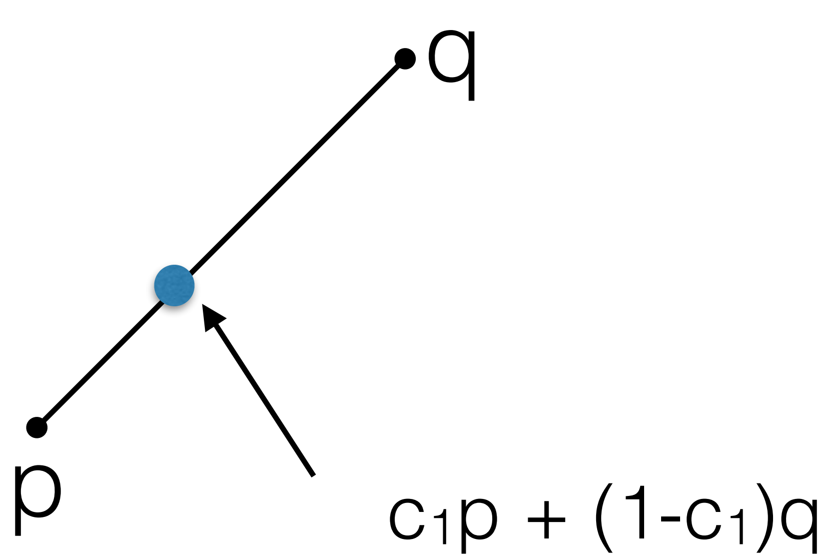
Outline

- Properties of CH
- Algorithms for computing the CH (P)
    - Brute-force
    - Gift wrapping (or: Jarviz march)
    - Quickhull
    - Graham scan
    - Andrew’s monotone chain
    - Incremental
    - Divide-and-conquer
- Can we do better?
    - Lower bound for CH

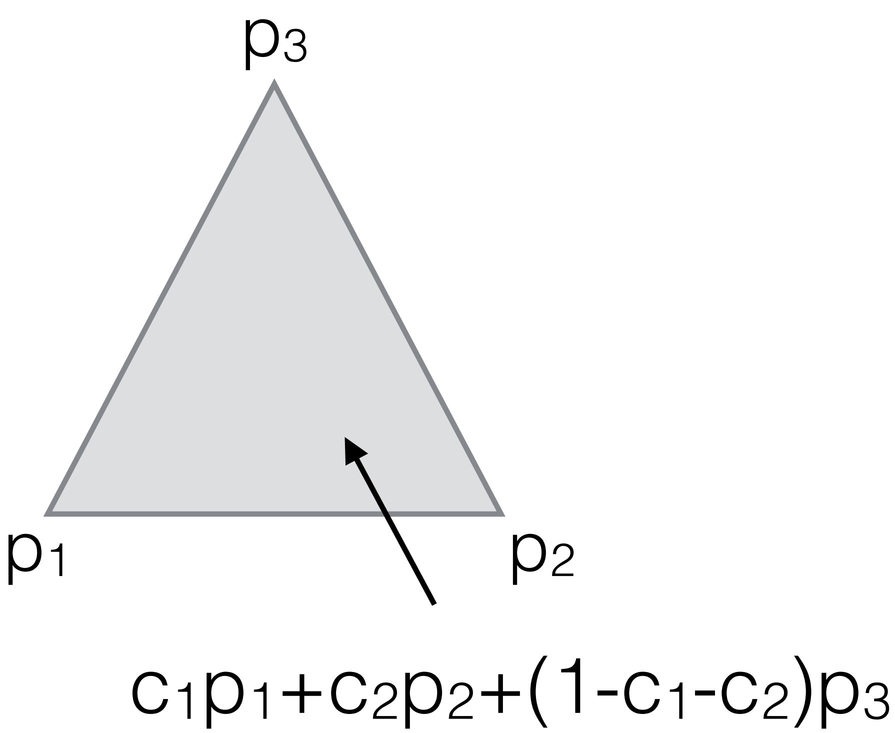
Convexity: algebraic view

- A **convex combination** of points p_1, p_2, …p_k is a point of the form

$$c_1p_1+c_2p_2+\ldots c_kp_k, \text{ with } c_i \text{ in } [0,1],\ c_1+c_2+\ldots+c_k=1$$



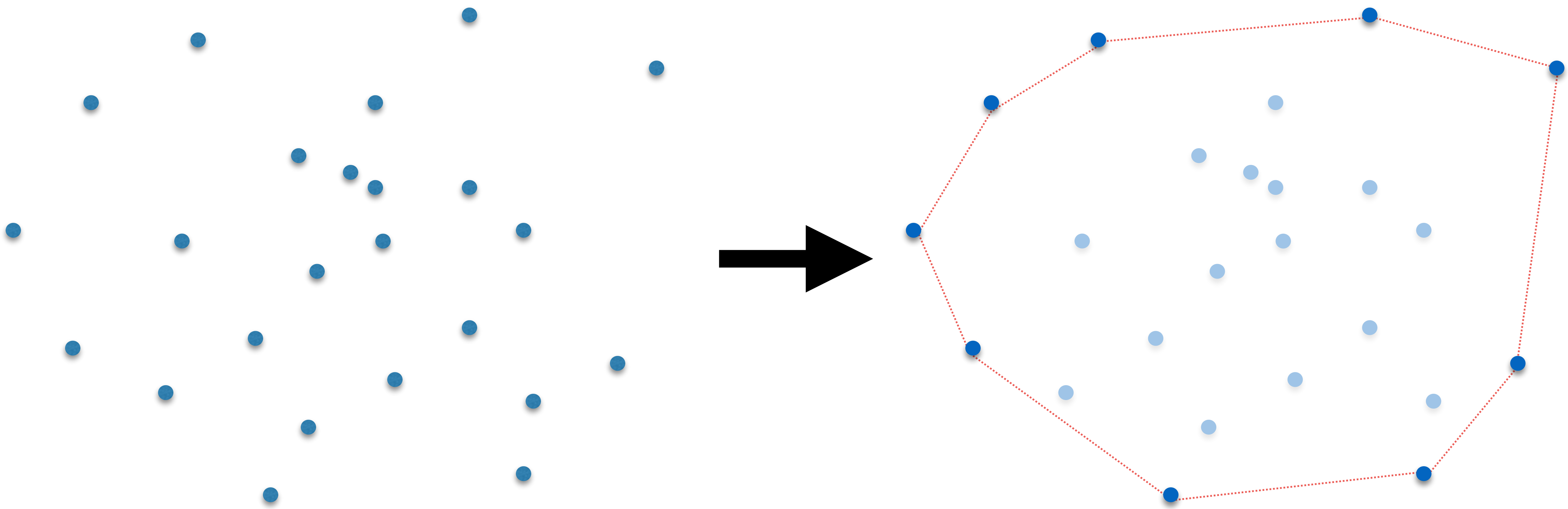
a segment consists of all convex combinations of its 2 vertices



a triangle consists of all convex combinations of its 3 vertices

- The convex hull CH(P) = all convex combinations of points in P

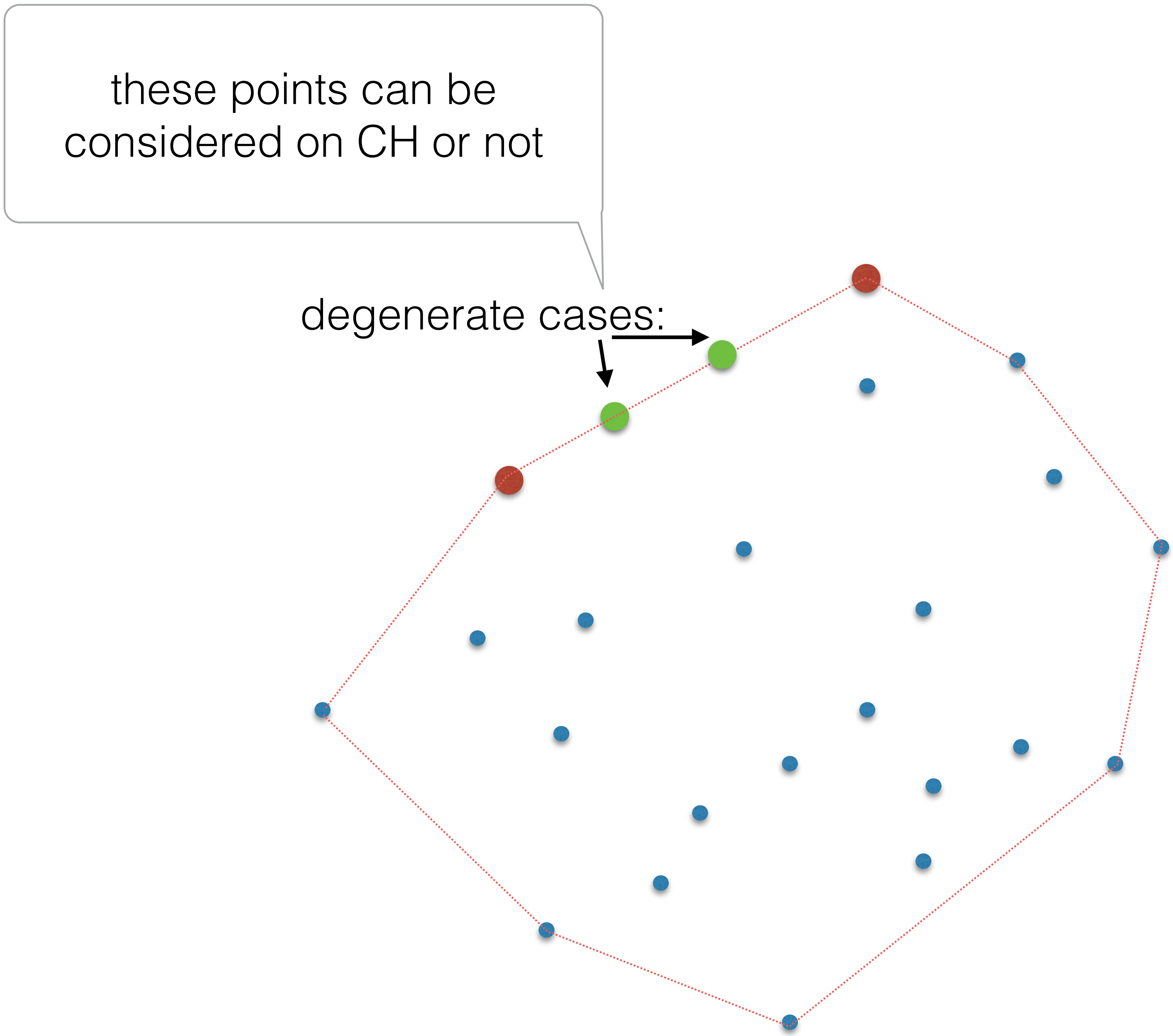
Convex Hull



Input:
array P of points (in 2D)

Output:
list of points on the CH (in boundary order)

What exactly is on the CH?



Convex Hull Variants

- Several types of convex hull output are conceivable
  - **all** points on the convex hull in arbitrary order
  - **all** points on the convex hull in boundary order
  - **only non-collinear** points in arbitrary order
  - **only non-collinear** points in boundary order

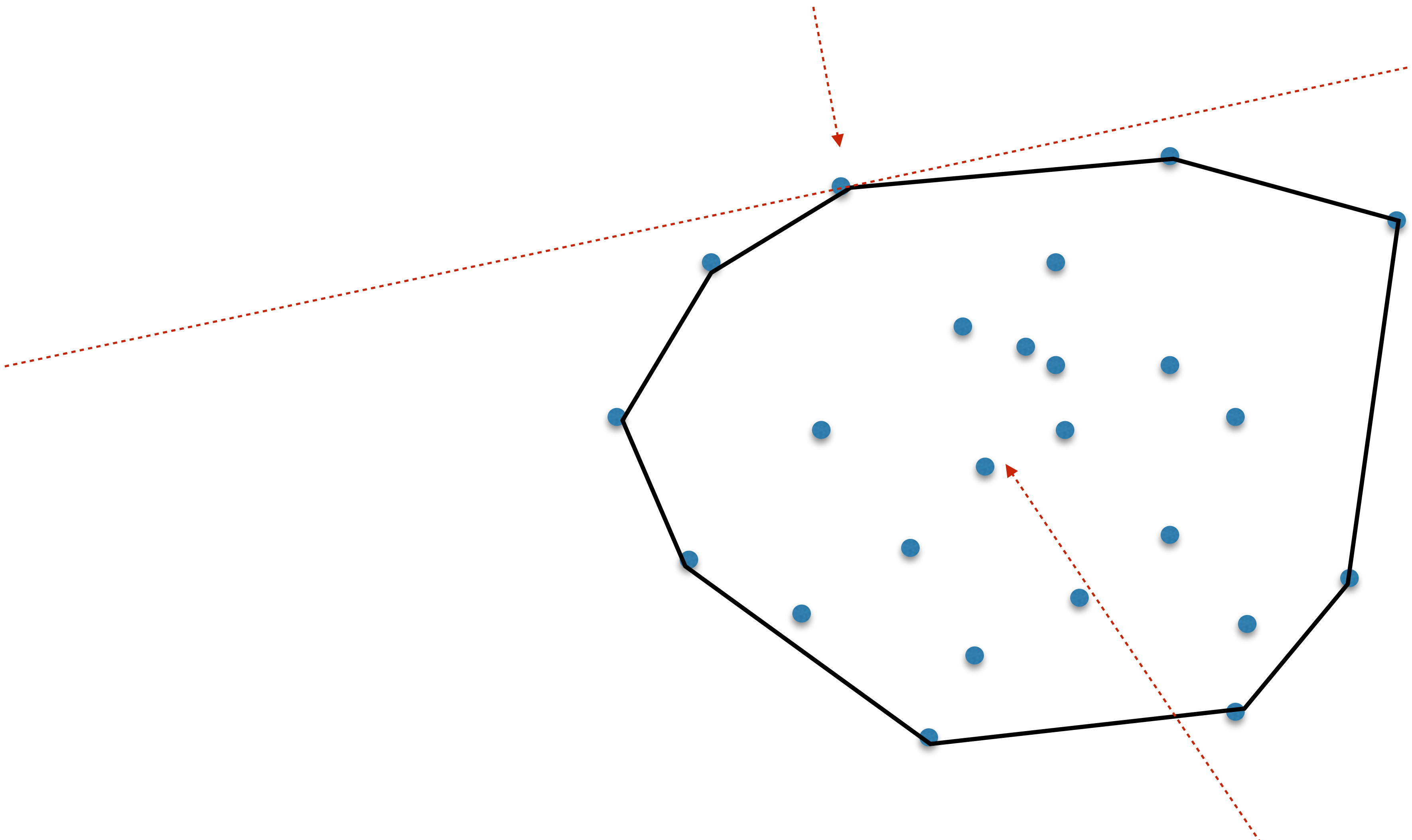
- It may seem that computing in boundary order is harder
  - we'll see that identifying the points on the CH is Omega(n lg n)

    ==> sorting is not dominant

Convex Hull:

Basic properties

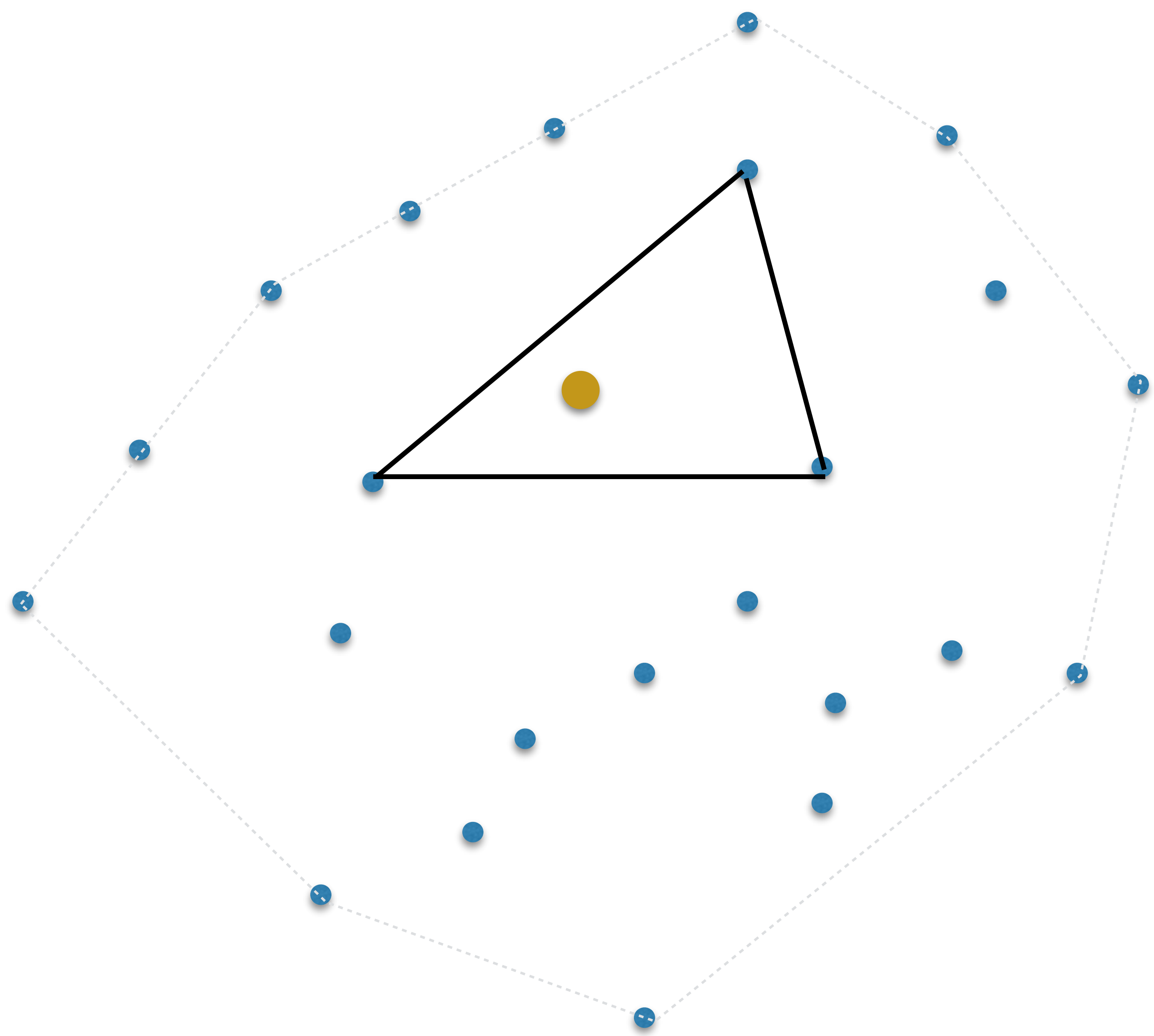
Points on the CH are extreme



Points **not** on the CH are interior

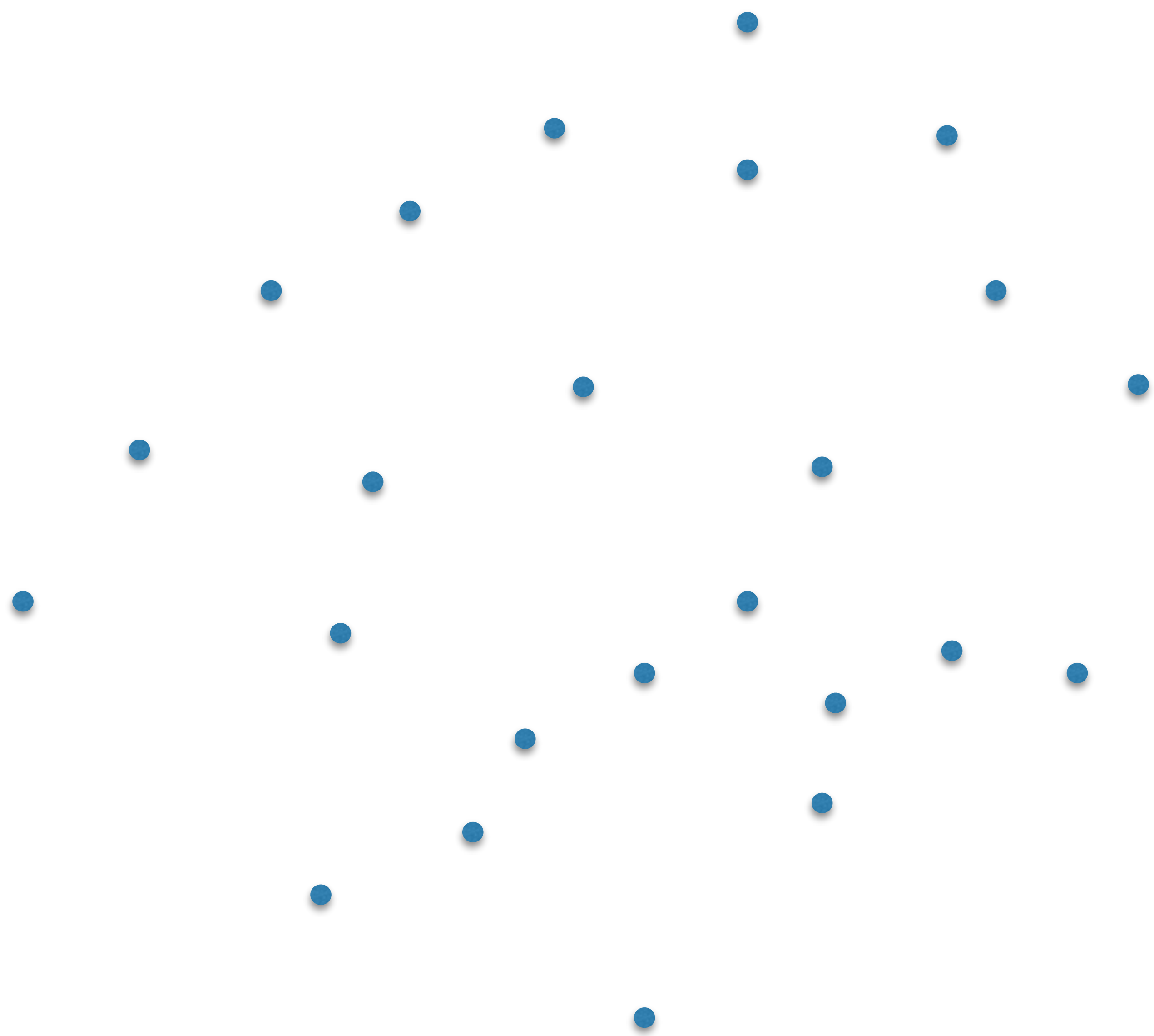
Interior points

- A point p is called interior if p is contained in the interior of a triangle formed by three other points of P (or: in interior of a segment formed by two points).
- Claim: p interior <==> p **not** on the CH



Extreme points

- A point p is called **extreme** if there exists a line l through p, such that all the other points of P are on the same side of l (or on l)



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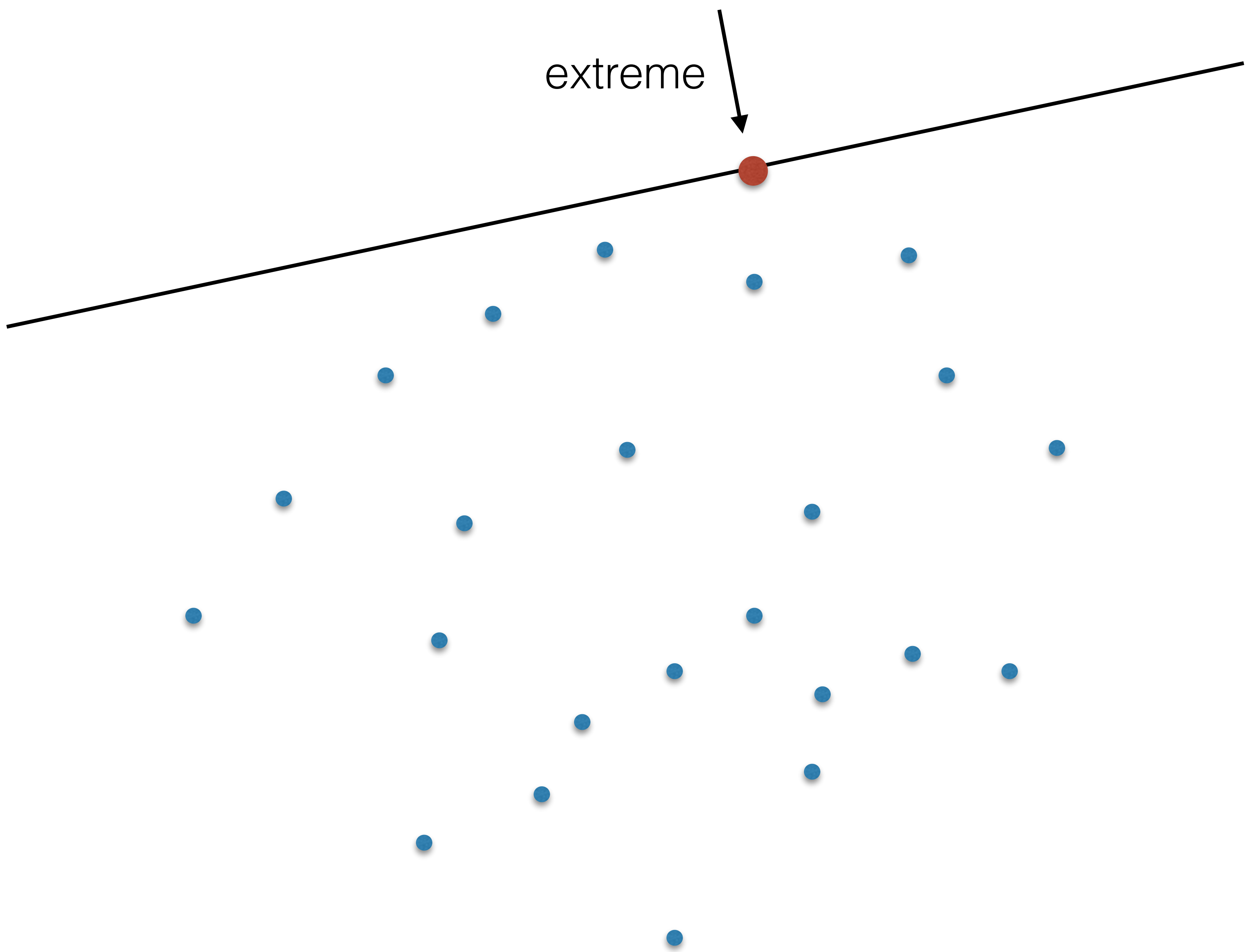

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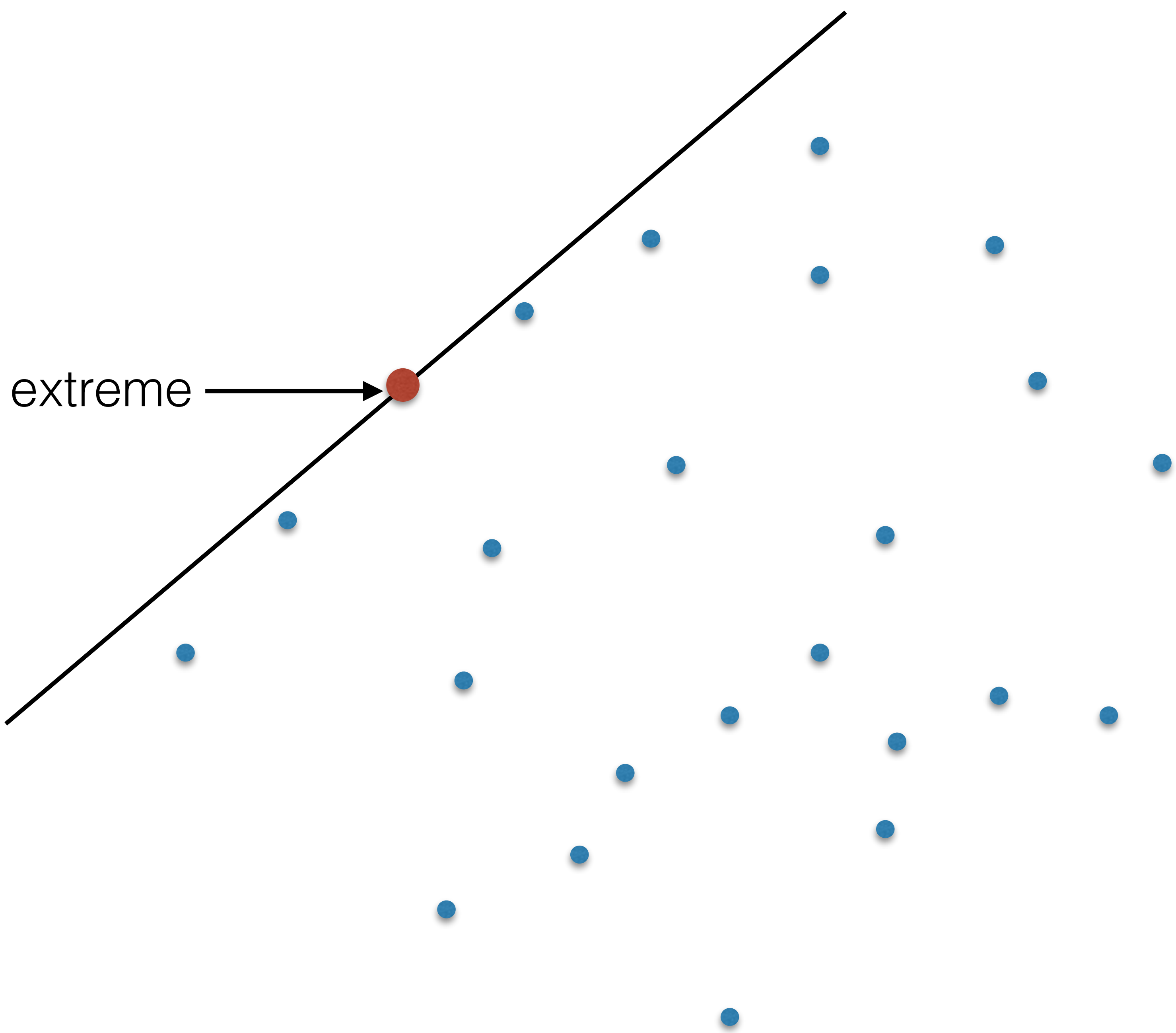

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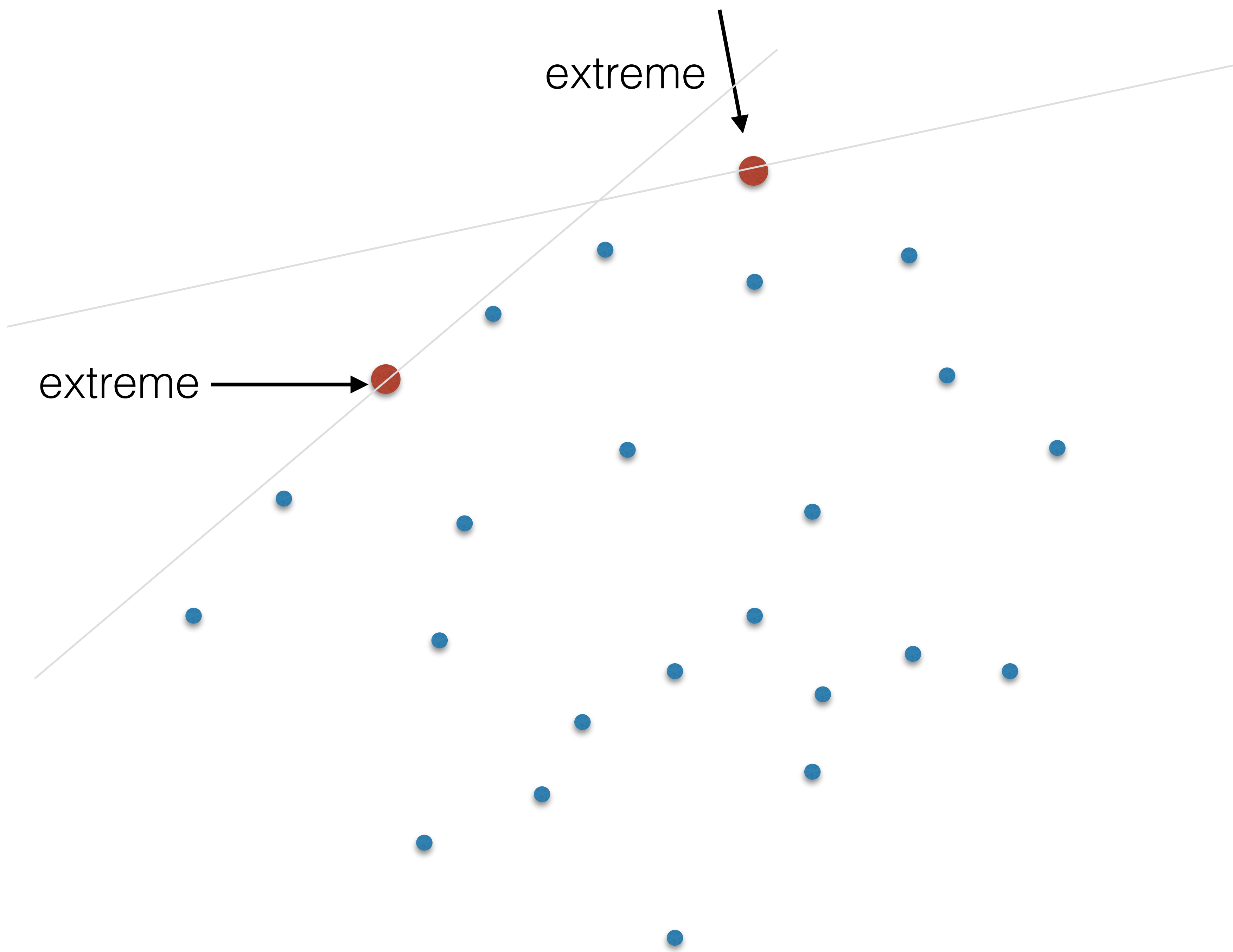

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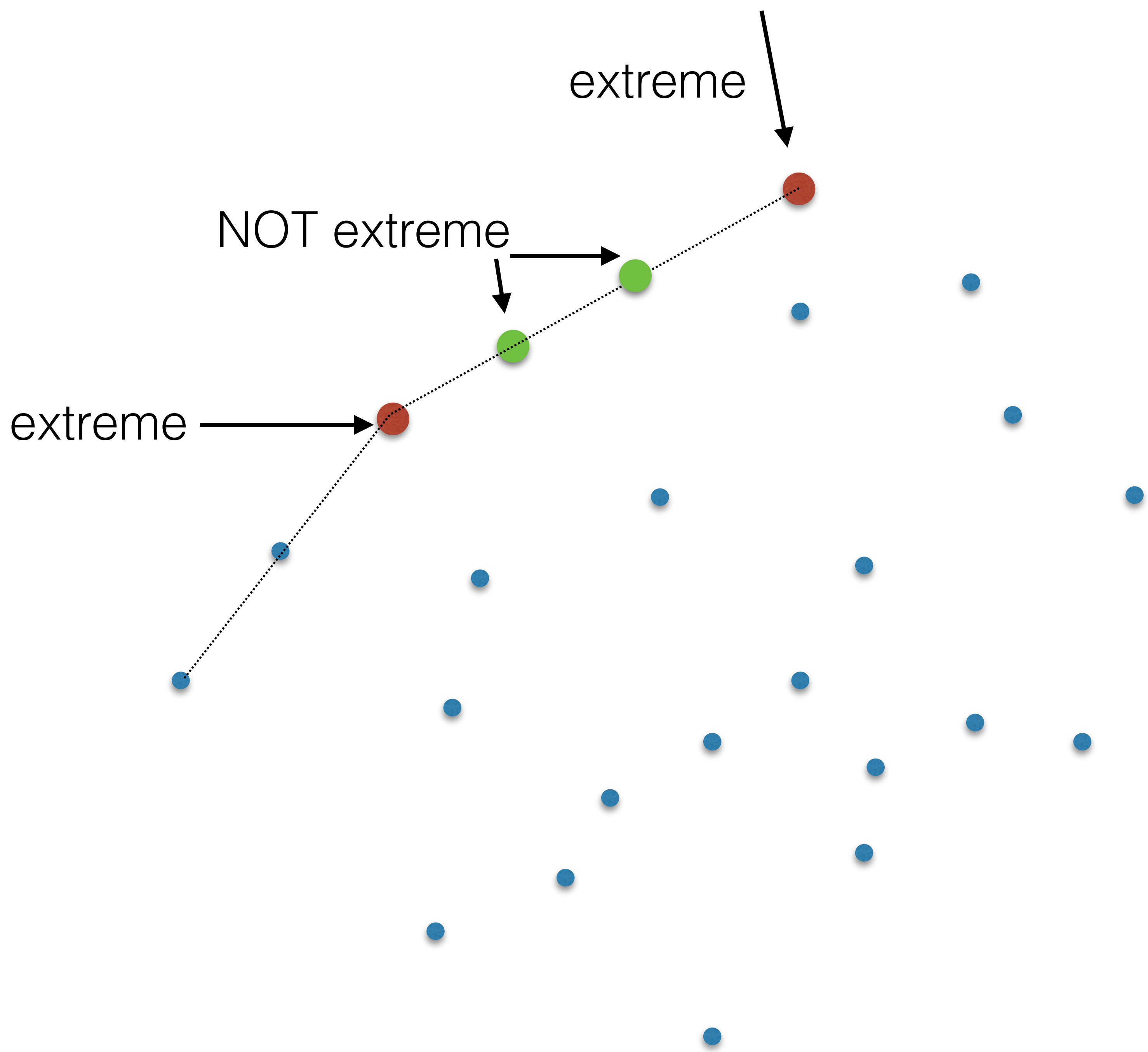

Extreme points

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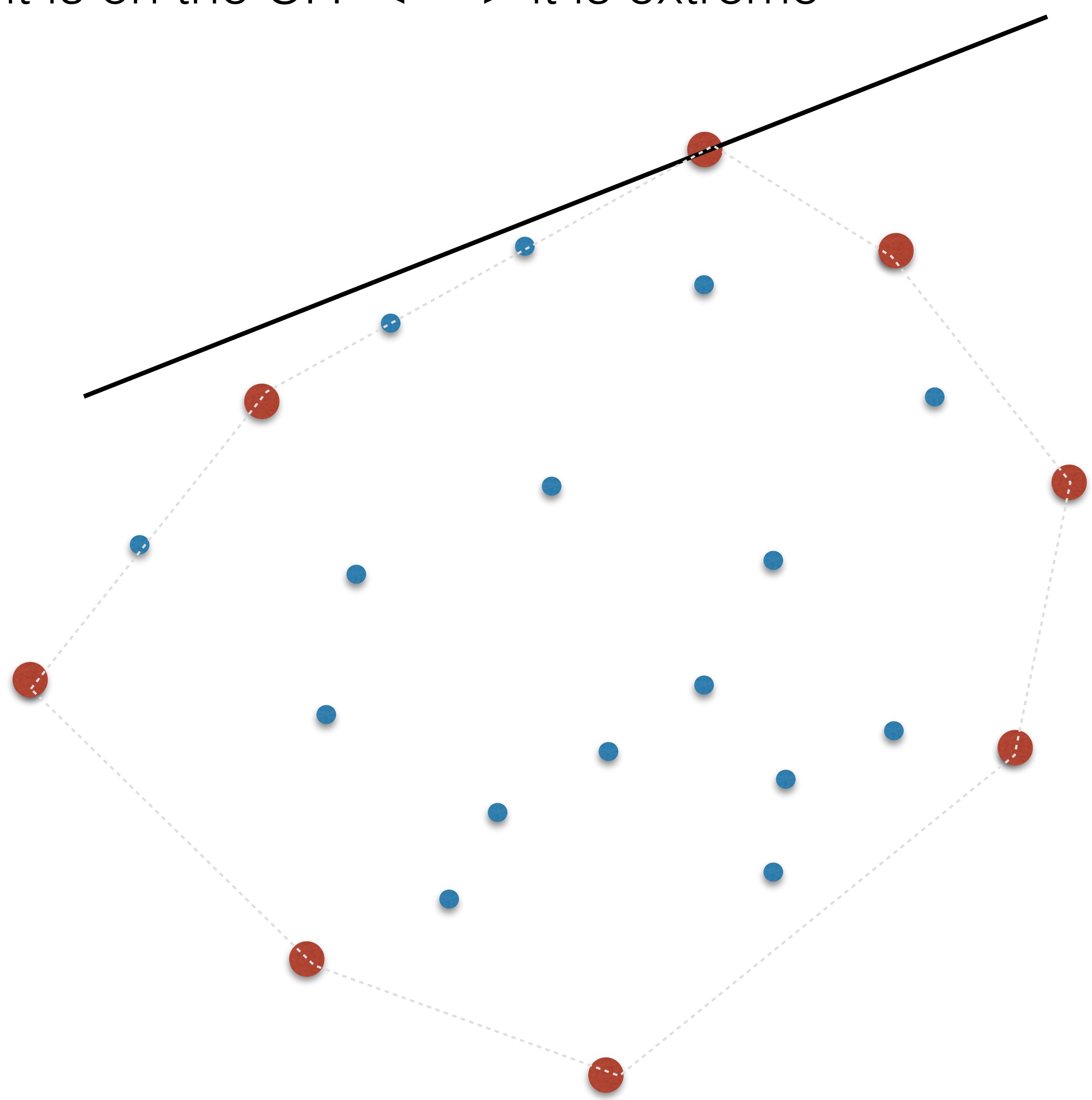
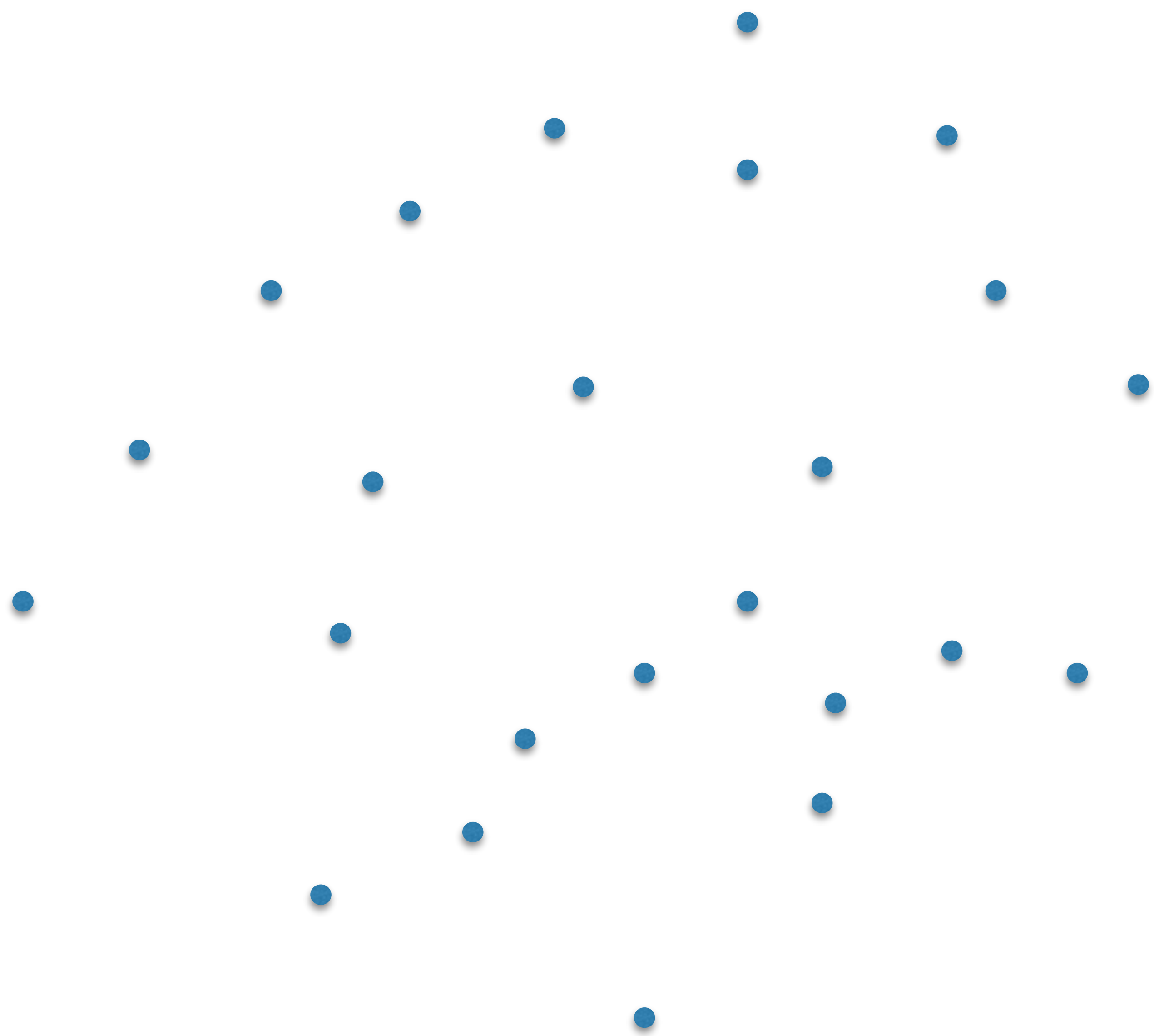
- Claim: A point is on the CH <==> it is extreme



Extreme edges

- An edge (p_i, p_j) is extreme if all the other points of P are on one side of it (or on)



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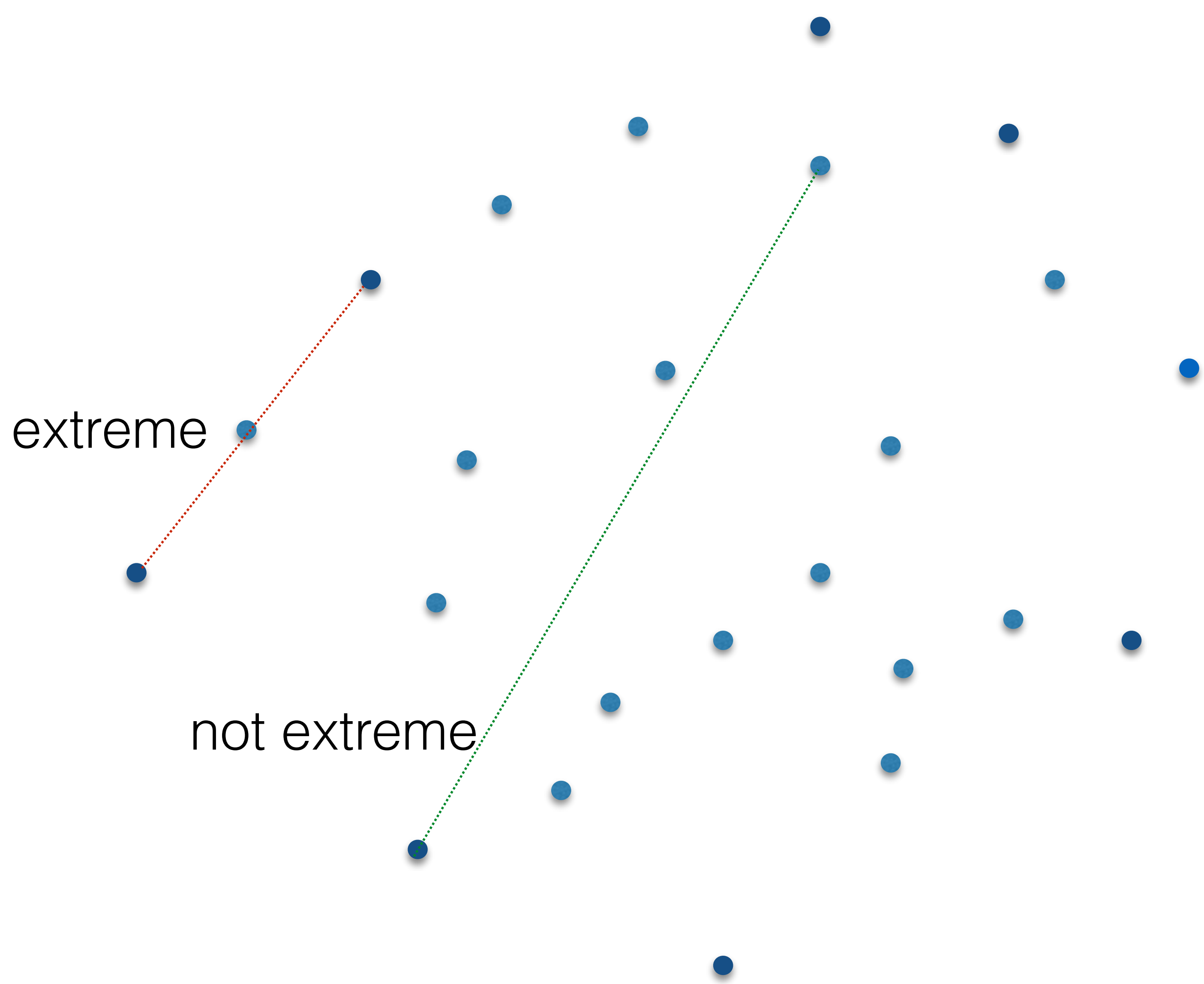
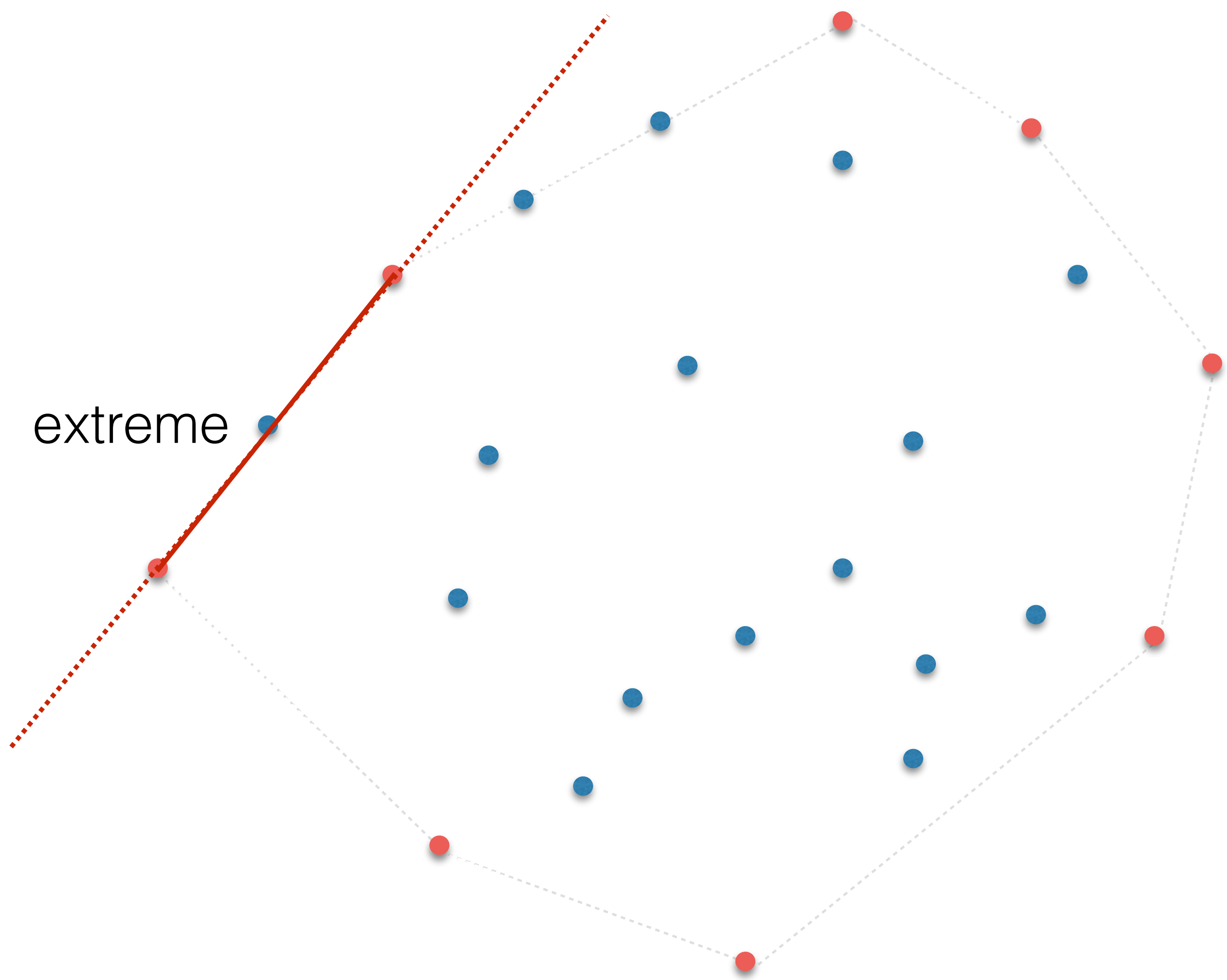


Extreme edges

- An edge (p_i, p_j) is extreme if all the other points of P are on one side of it (or on)
- Claim: A pair of points (p_i, p_j) form an edge on the CH iff edge (p_i, p_j) is extreme.



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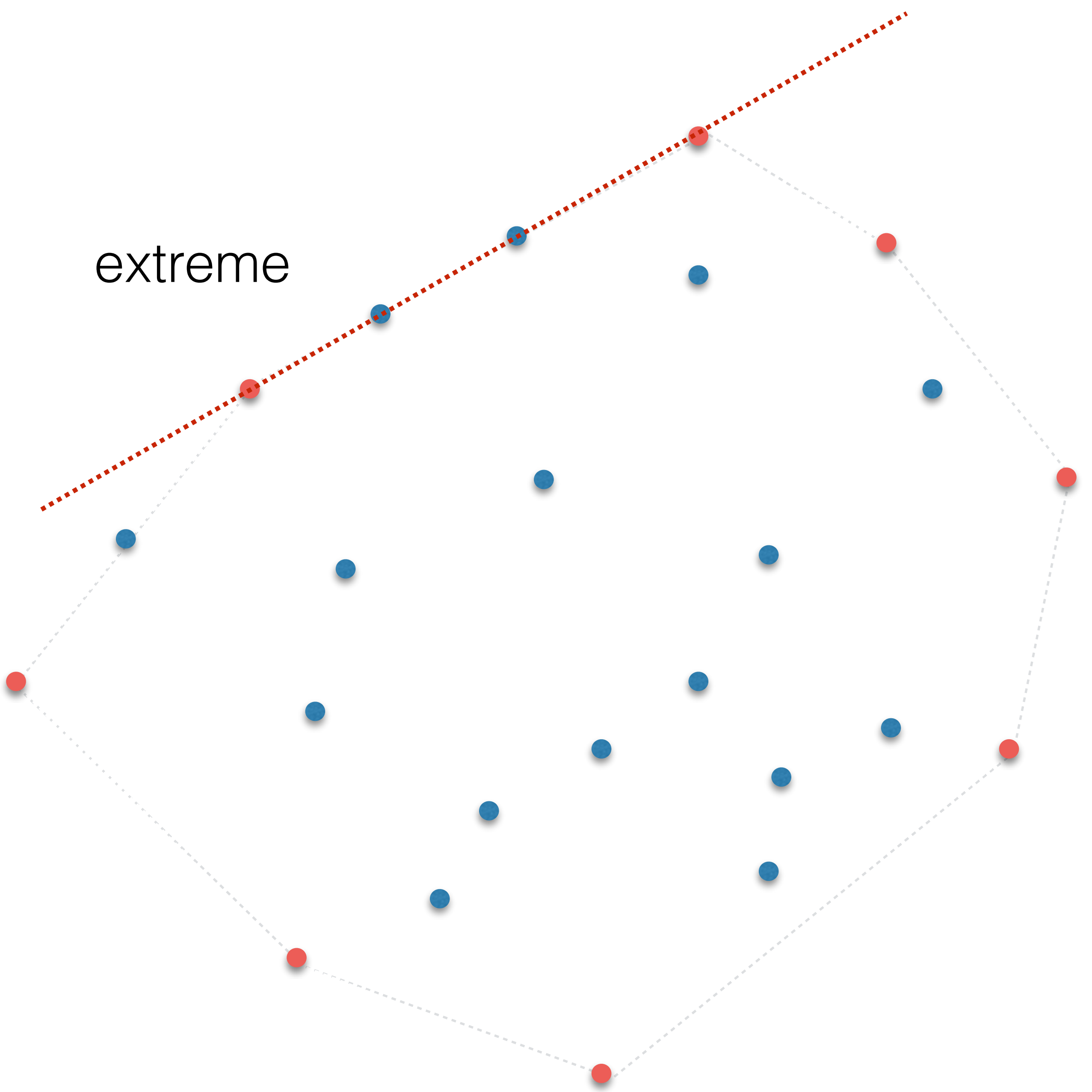

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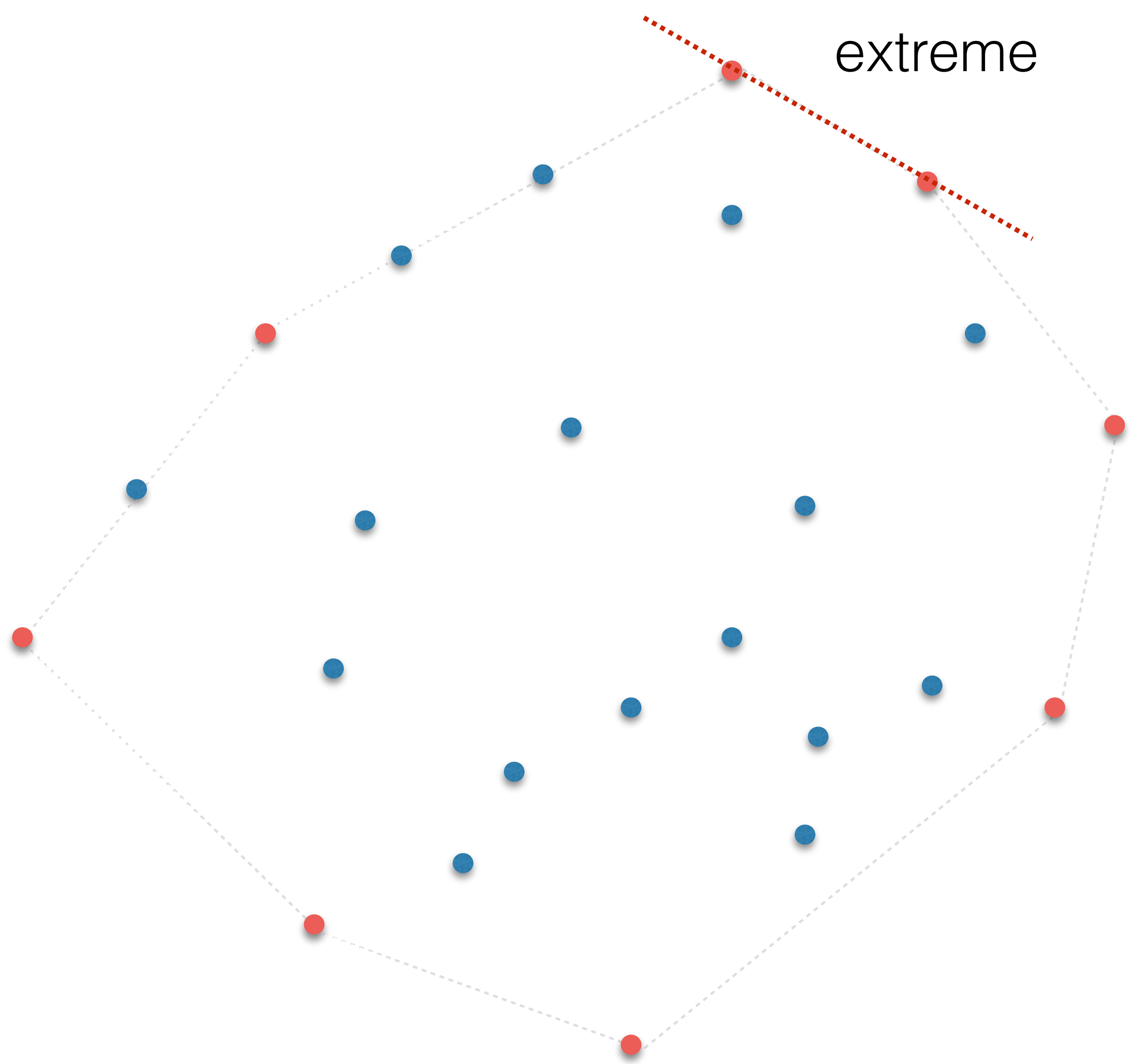

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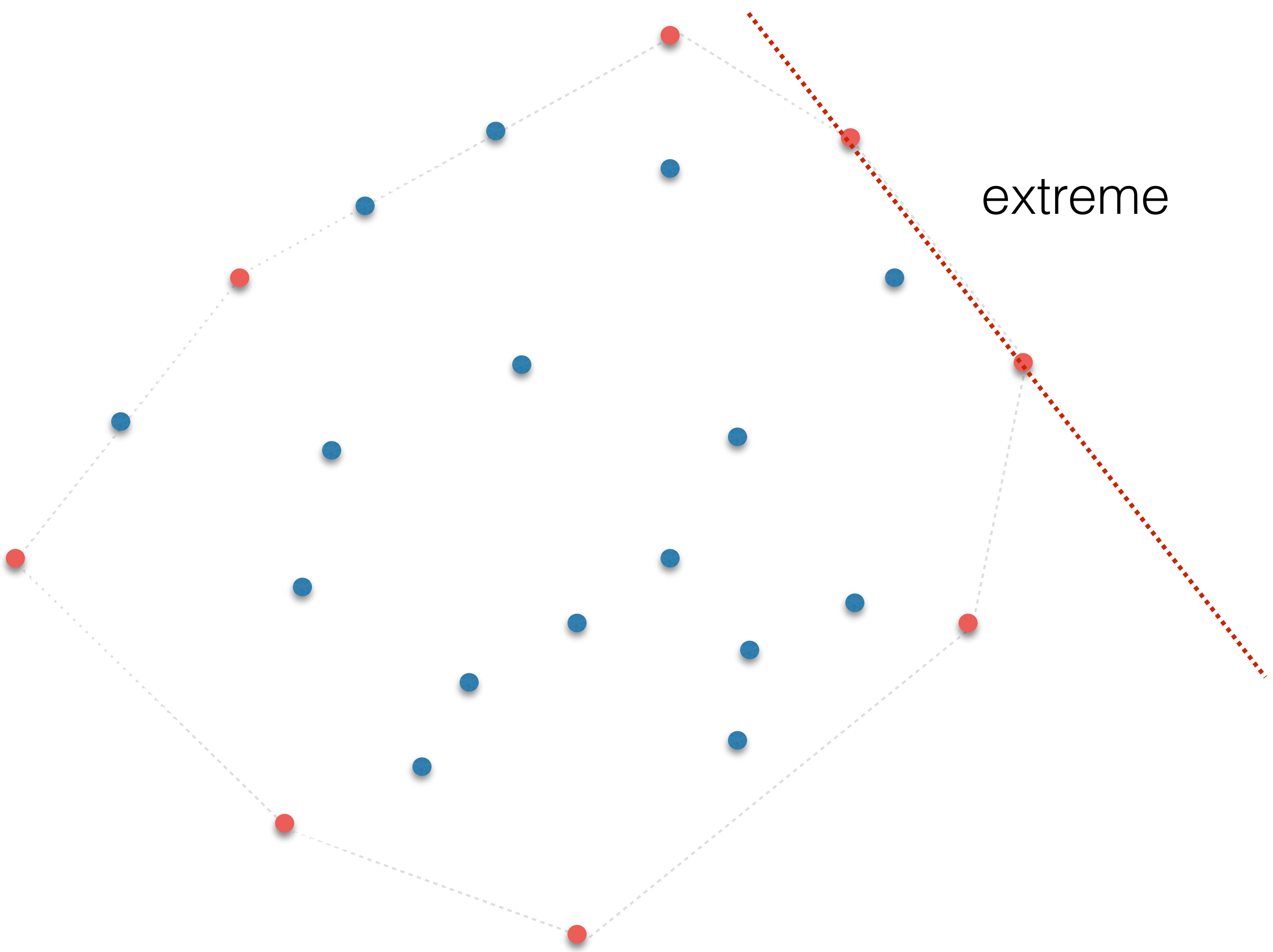

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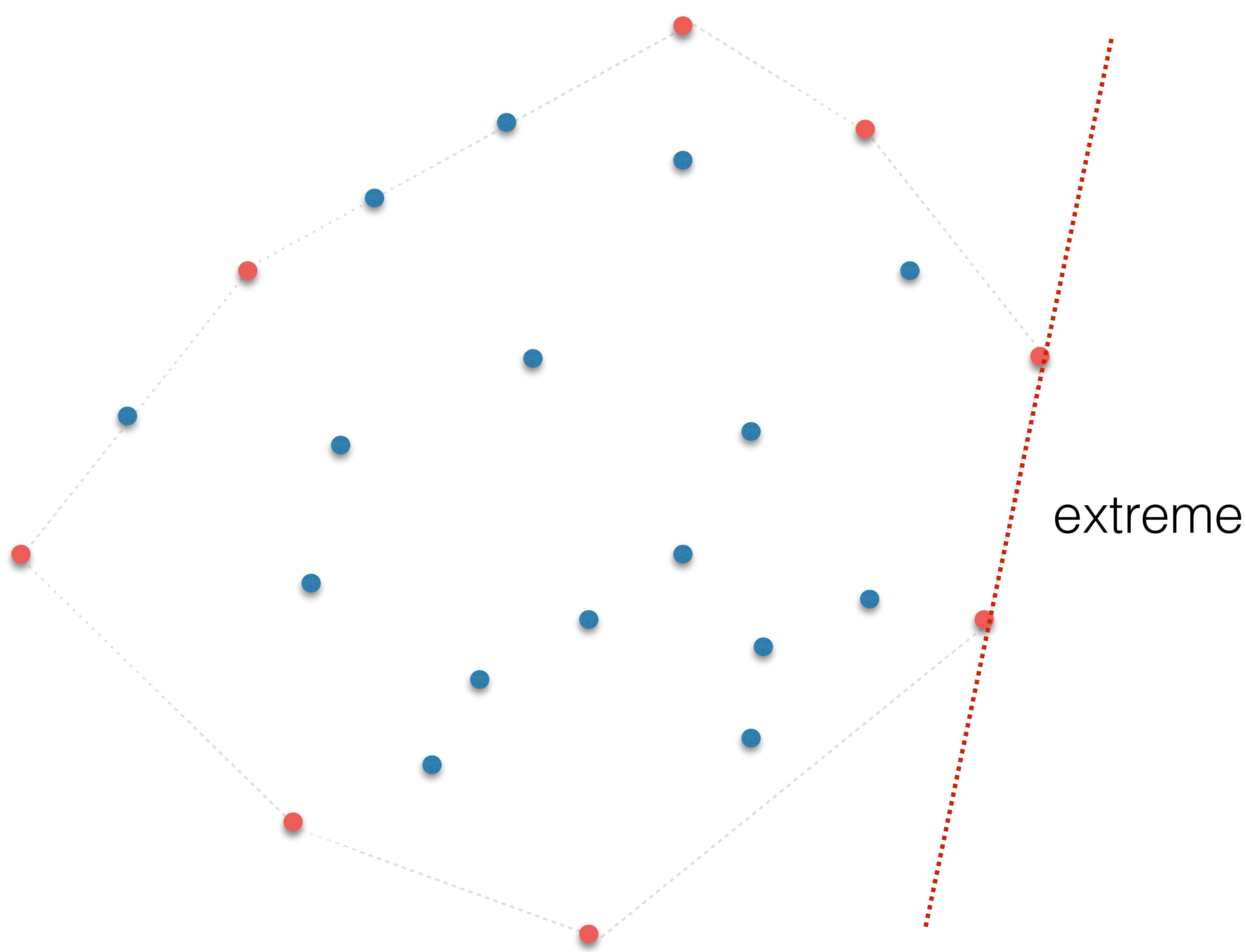

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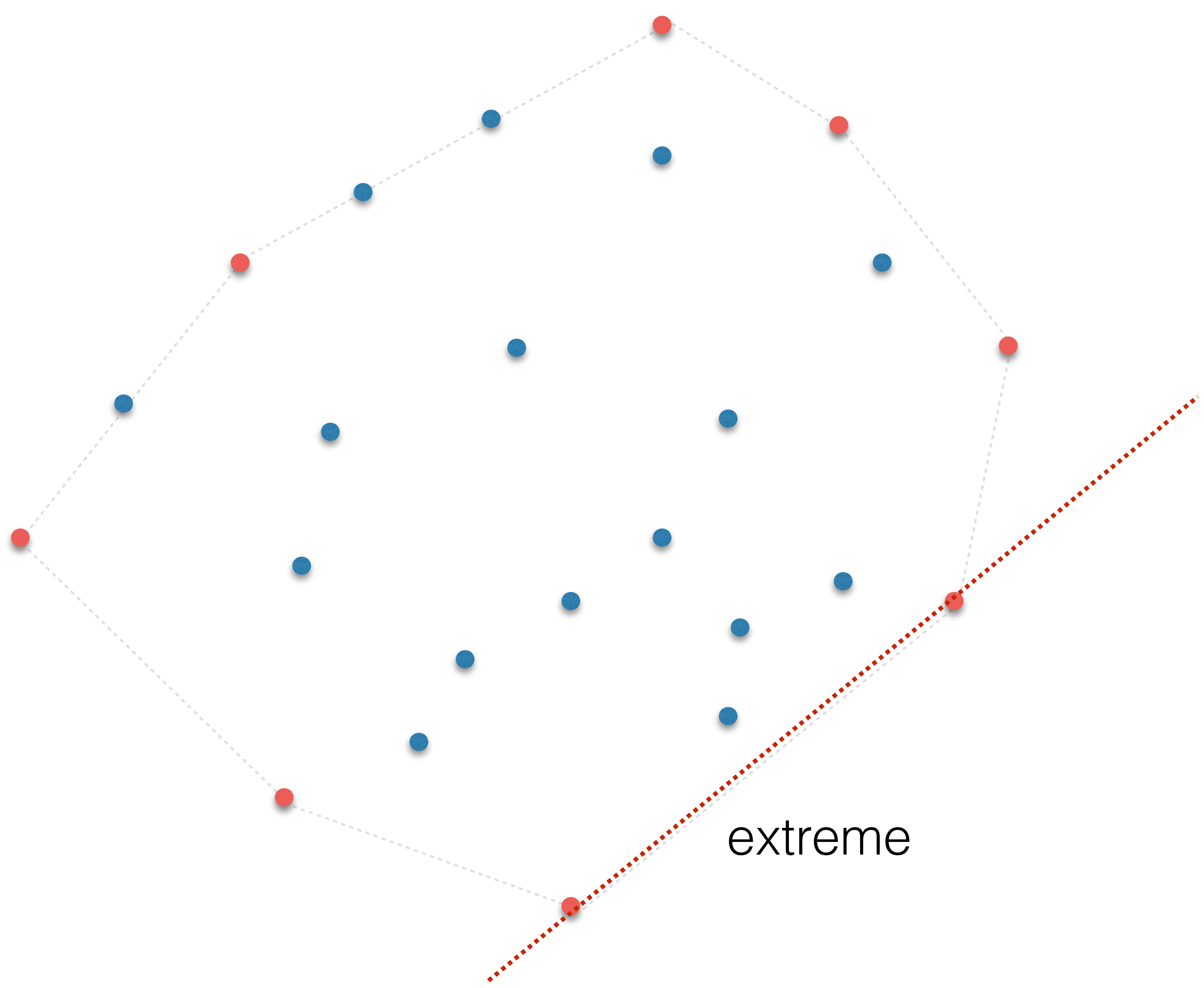


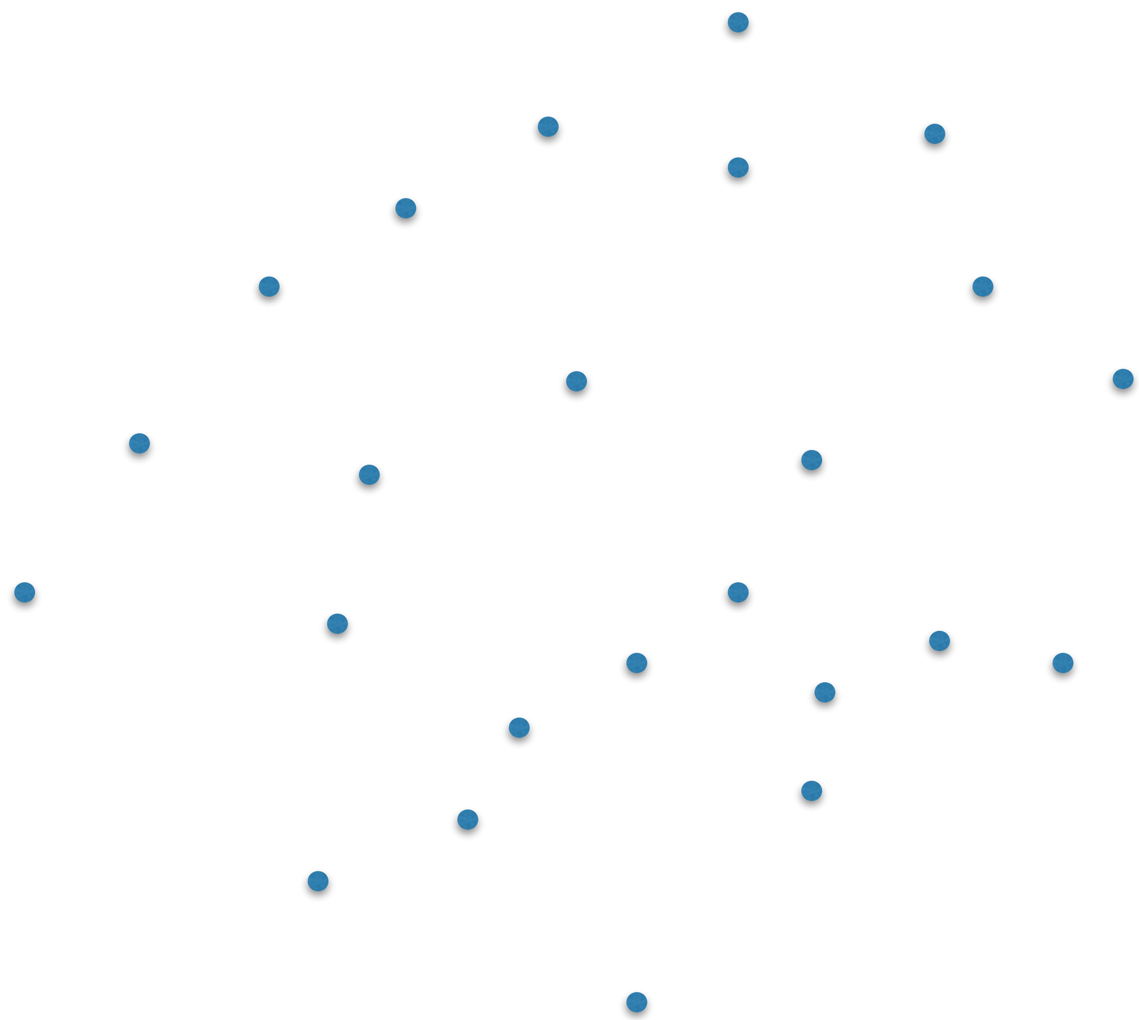
Summary

- CH consists of extreme points and edges!
    - point p is interior <==> p **not** on the CH
    - point p is extreme <==> p on the CH
    - A pair of points (p_i, p_j) form an edge on the CH <==> edge (p_i, p_j) is extreme
- First algorithm idea: find the CH by testing which edges are extreme

Brute force: Find extreme edges

Algorithm (input P)

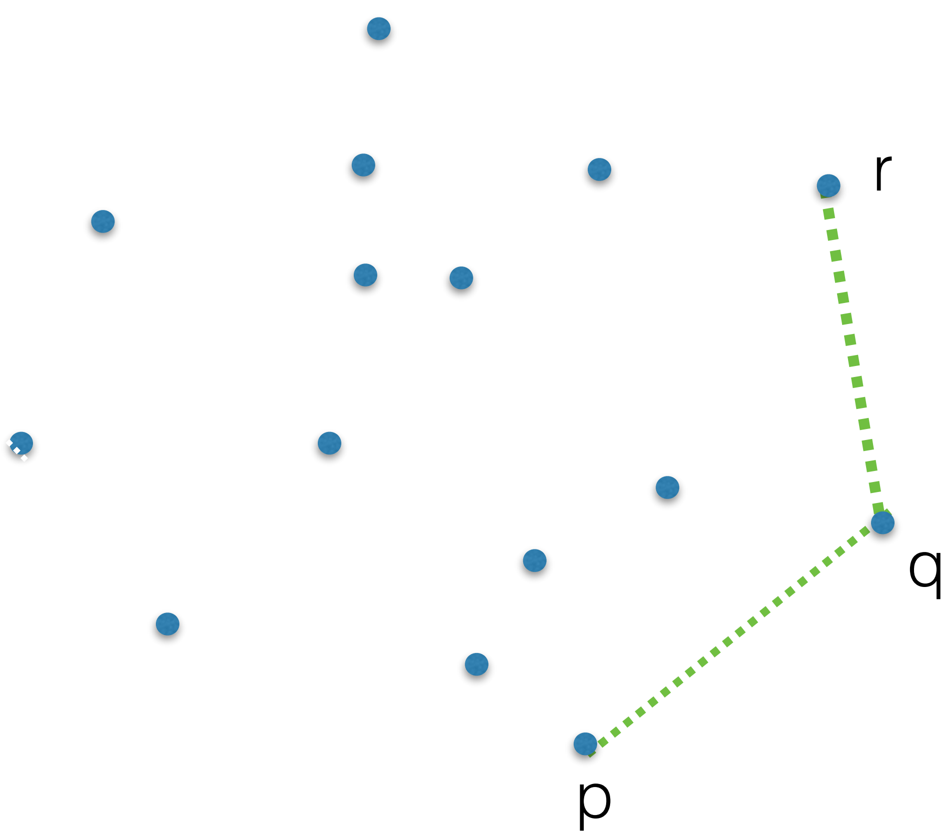
- for all distinct pairs (p_i, p_j)
  - check if edge (p_i,p_j) is extreme



- Analysis?

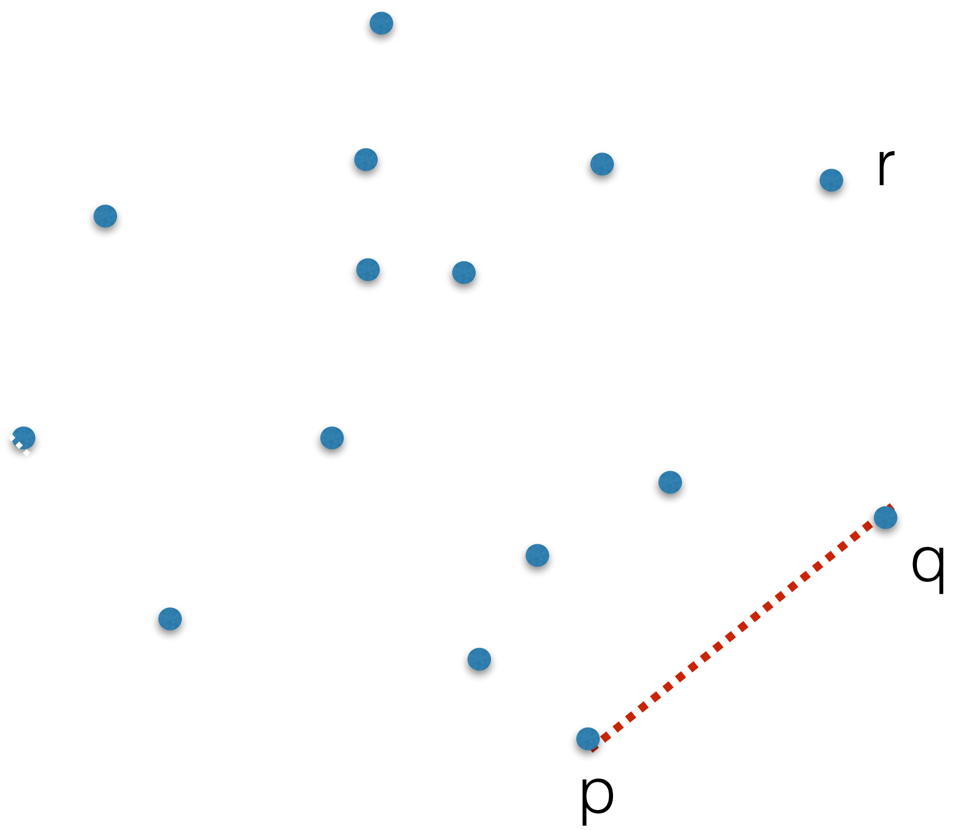
Gift wrapping (1970)

- Observation: CH consists of extreme edges, and each edge shares a vertex with next edge



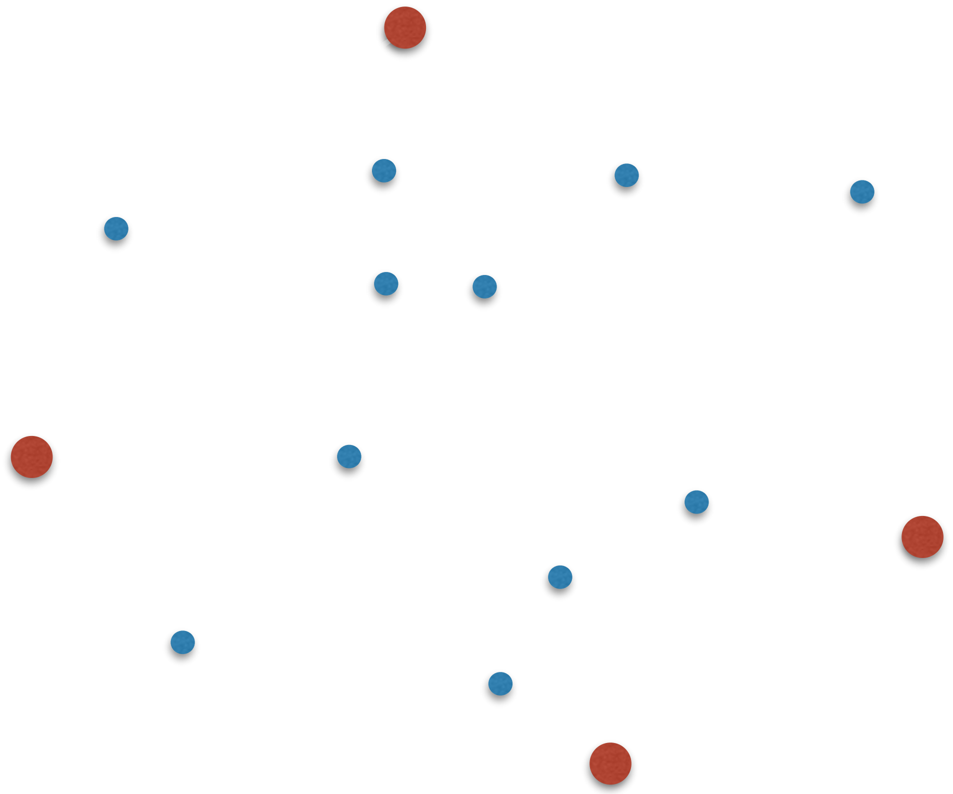
- Idea: use an edge to find the next one
    - How to find an extreme edge to start from?
    - Given an extreme edge, how to find the next one?

How to find an extreme edge to start from?

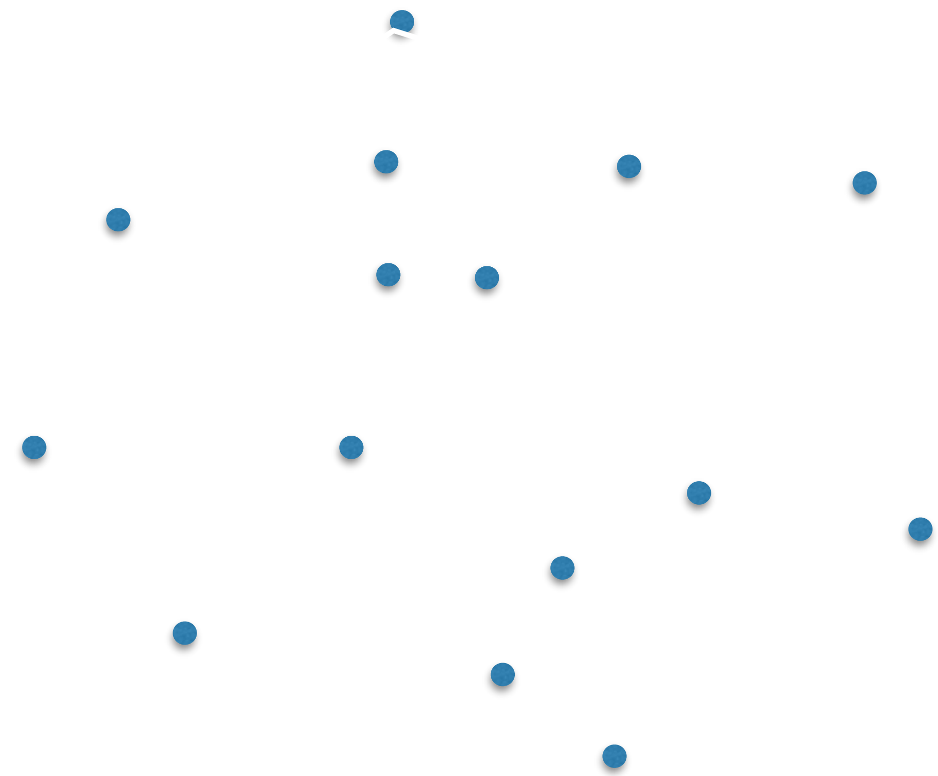


Can you think of some points that are guaranteed to be in CH?

- Claim
    - point with minimum x-coordinate is extreme
    - point with maximum x-coordinate is extreme
    - point with minimum y-coordinate is extreme
    - point with maximum y-coordinate is extreme
- Can you justify why?

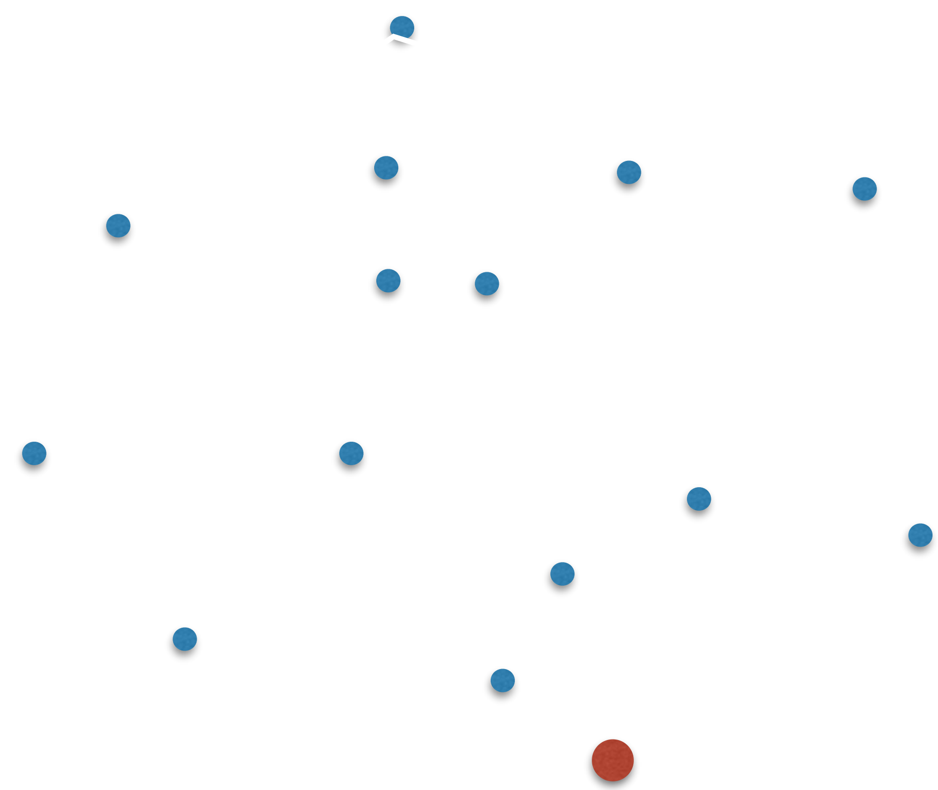


Gift wrapping (1970)



Gift wrapping (1970)

- Start from bottom-most point
    - if more then one, pick right most



Gift wrapping (1970)

- Start from bottom-most point
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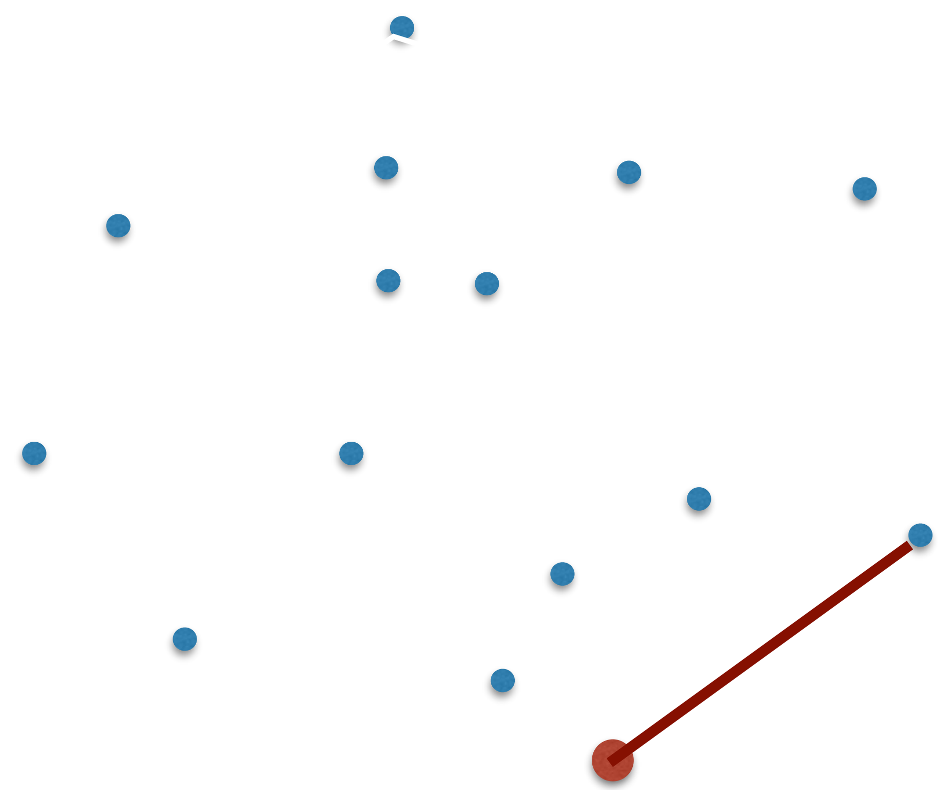
//find first edge. HOW ?



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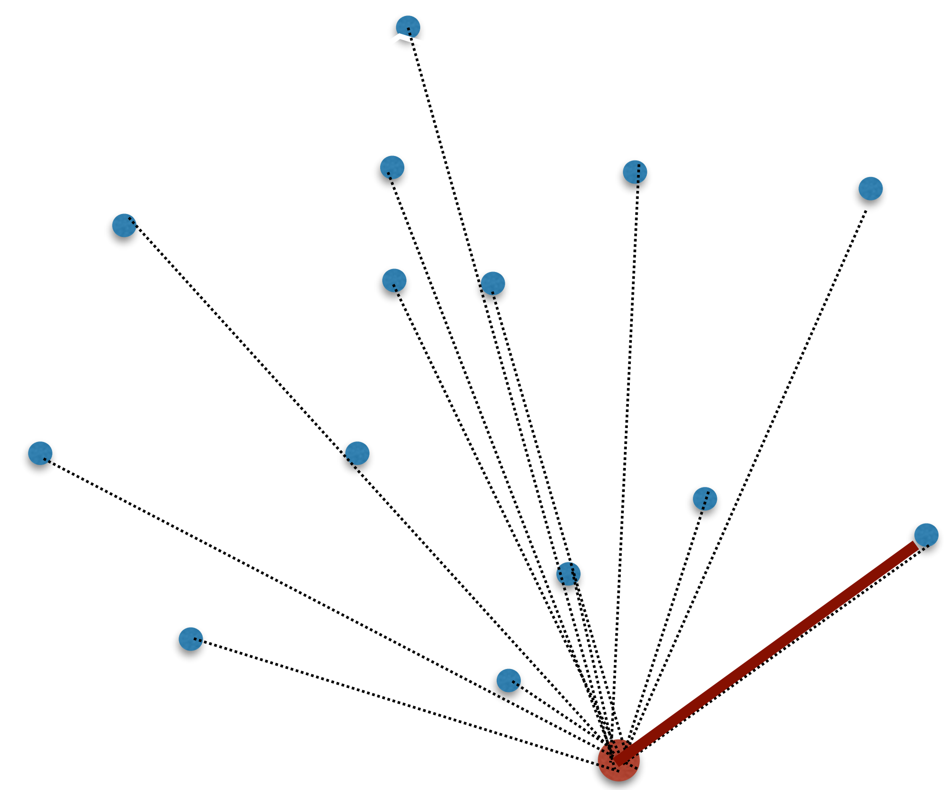

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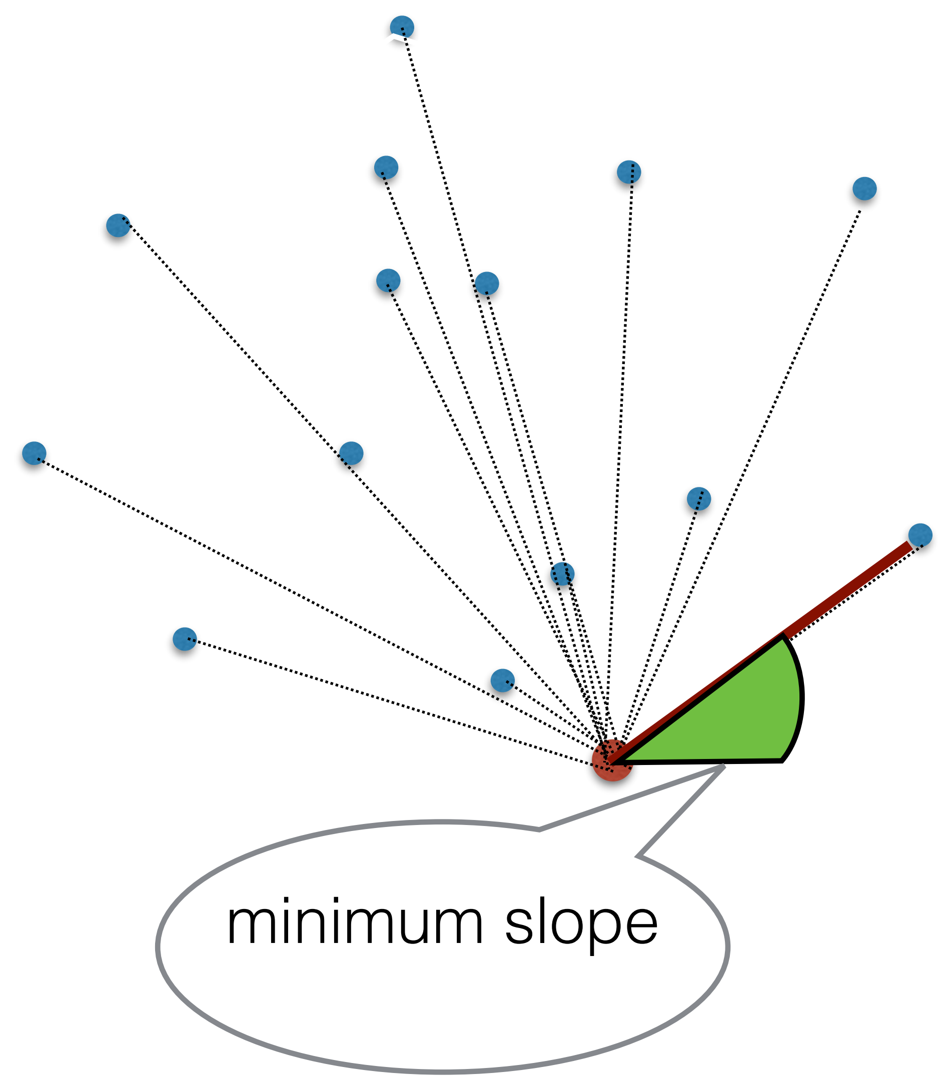


Gift wrapping (1970)

- Start from bottom-most point
    - if more then one, pick right most

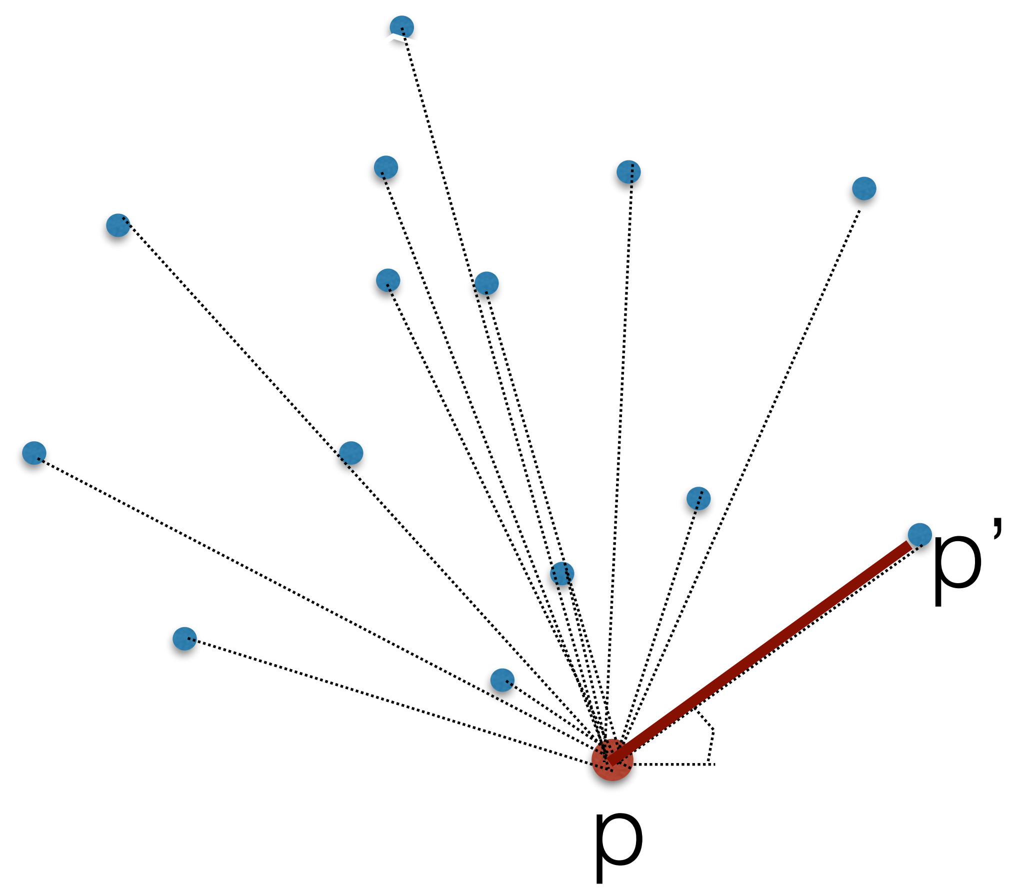
/******** find first edge ********/

- for each point q (q != p)
    - compute slope of q wrt p
- let p' = point with smallest slope

//claim: pp' is extreme edge

- output (p, p') as first edge

/*********what next ? ********/



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/******** find first edge ********/

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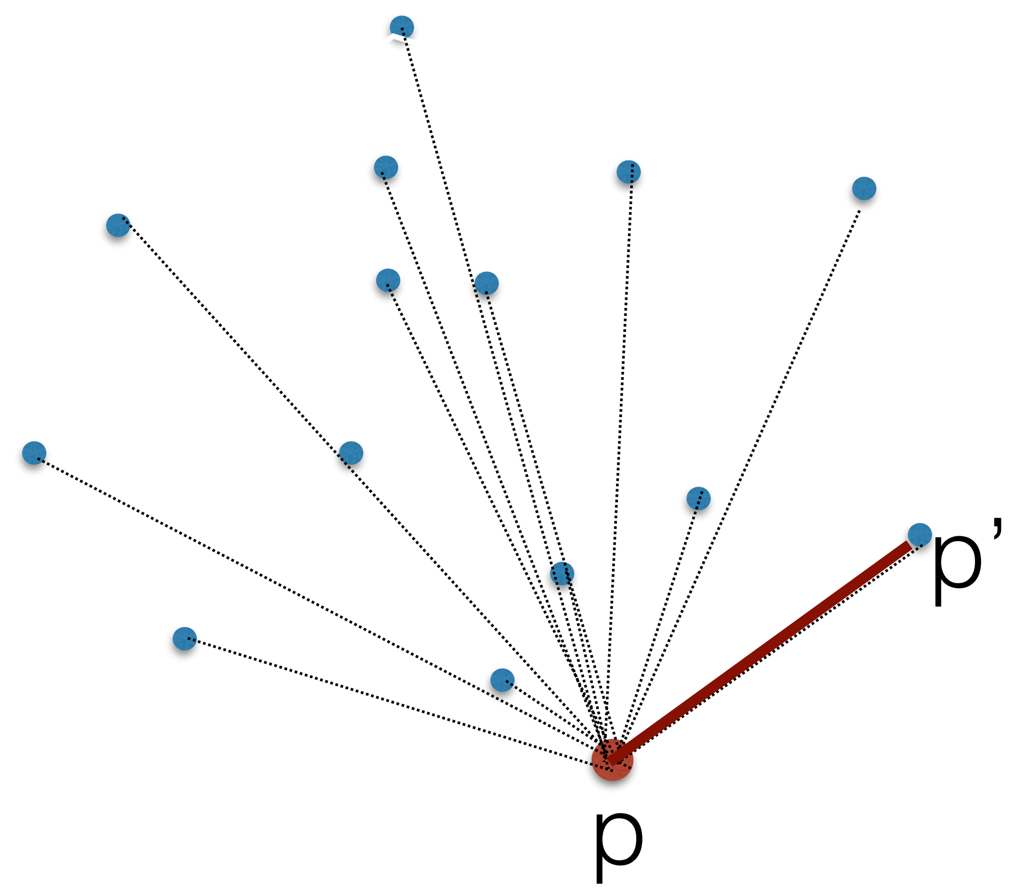
- let p' = point with smallest slope

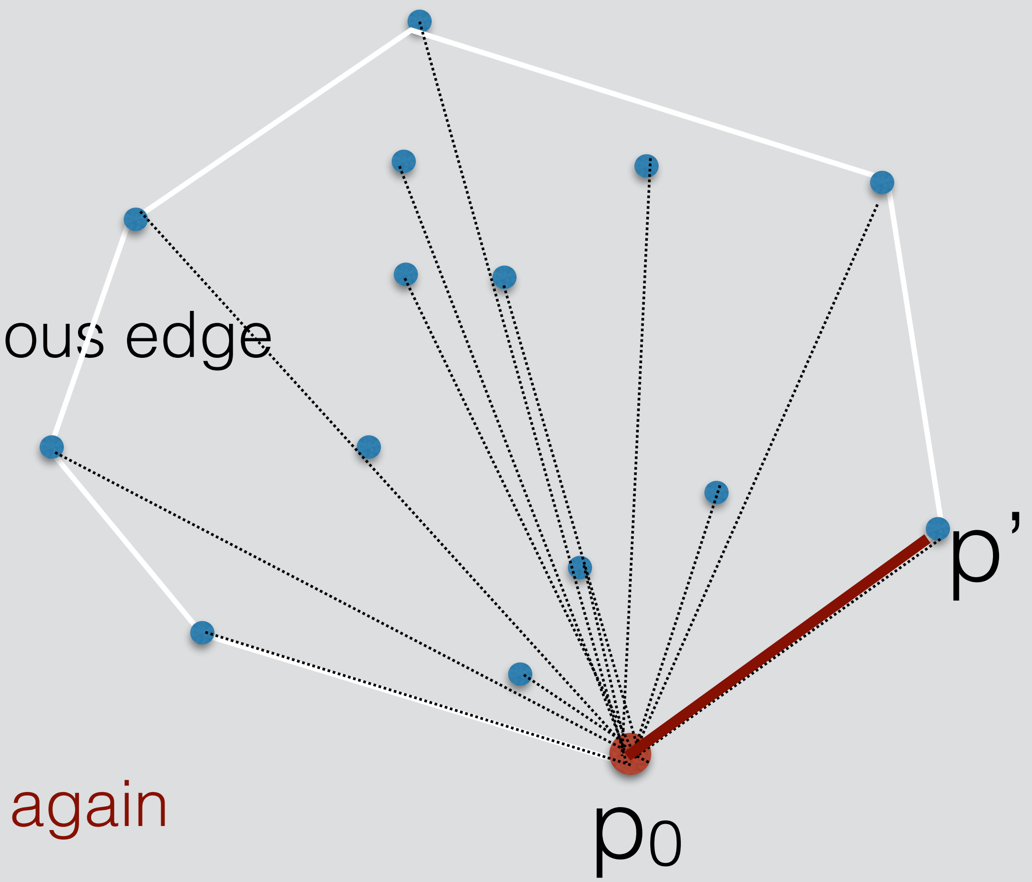
//claim: pp' is extreme edge

- output (p, p') as first edge
- repeat from p'



Gift wrapping (1970)

- p_0 = point with smallest y-coordinate (if more then one, pick right most)
- $p = p_0$
- repeat
  - for each point q (q != p)
    - compute ccw-angle of q wrt previous edge
  - let p' = point with smallest angle
  - output (p, p') as CH edge
  - p = p'
- until $p = p_0$ //until it discovers first point again



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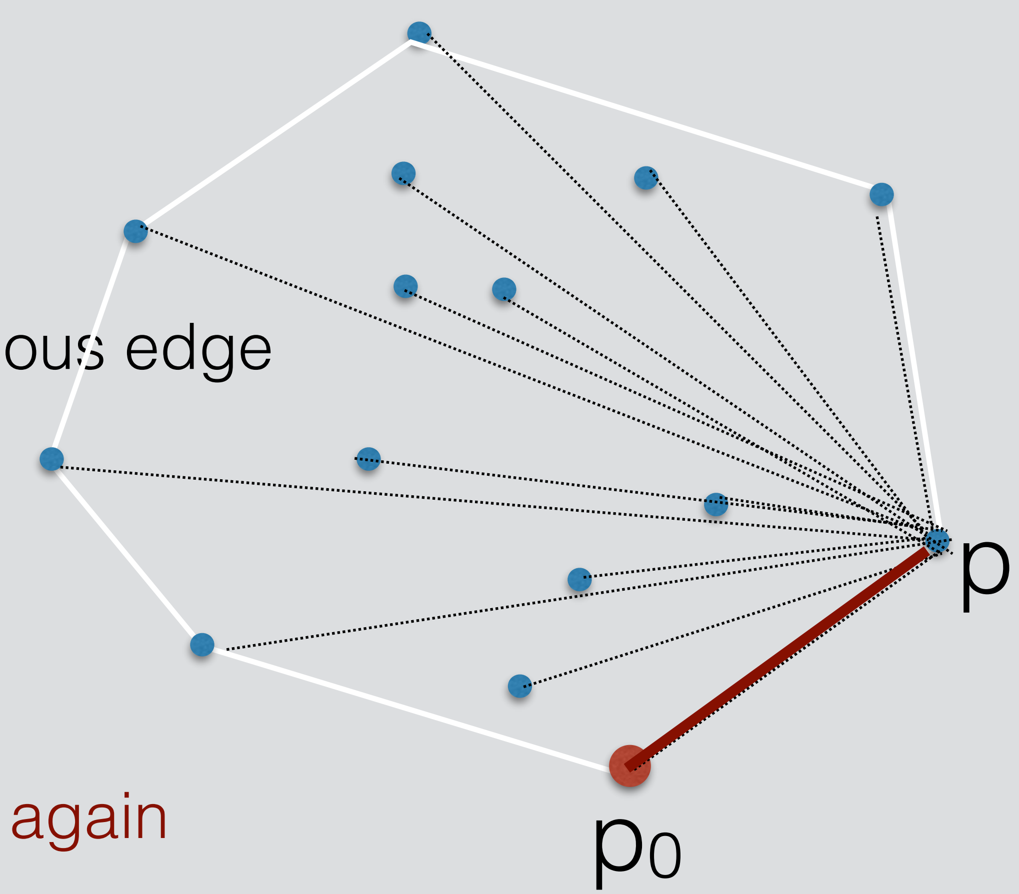

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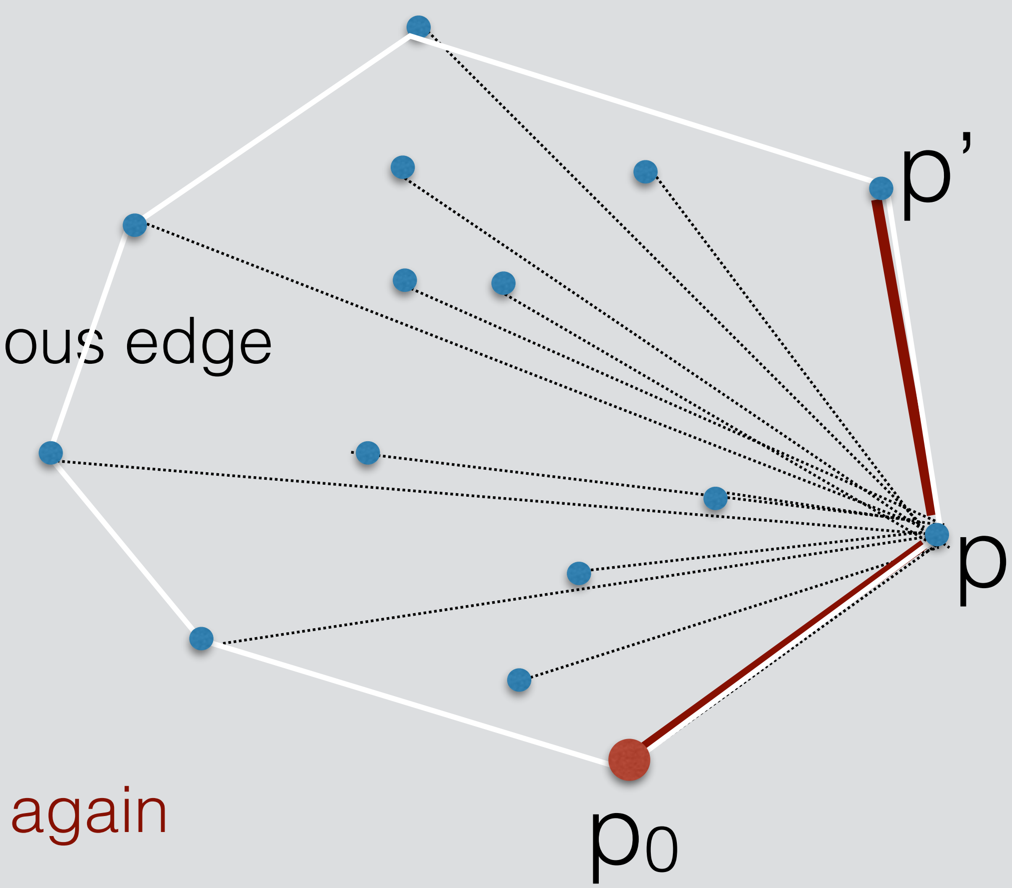


The gift wrapping algorithm : Classwork

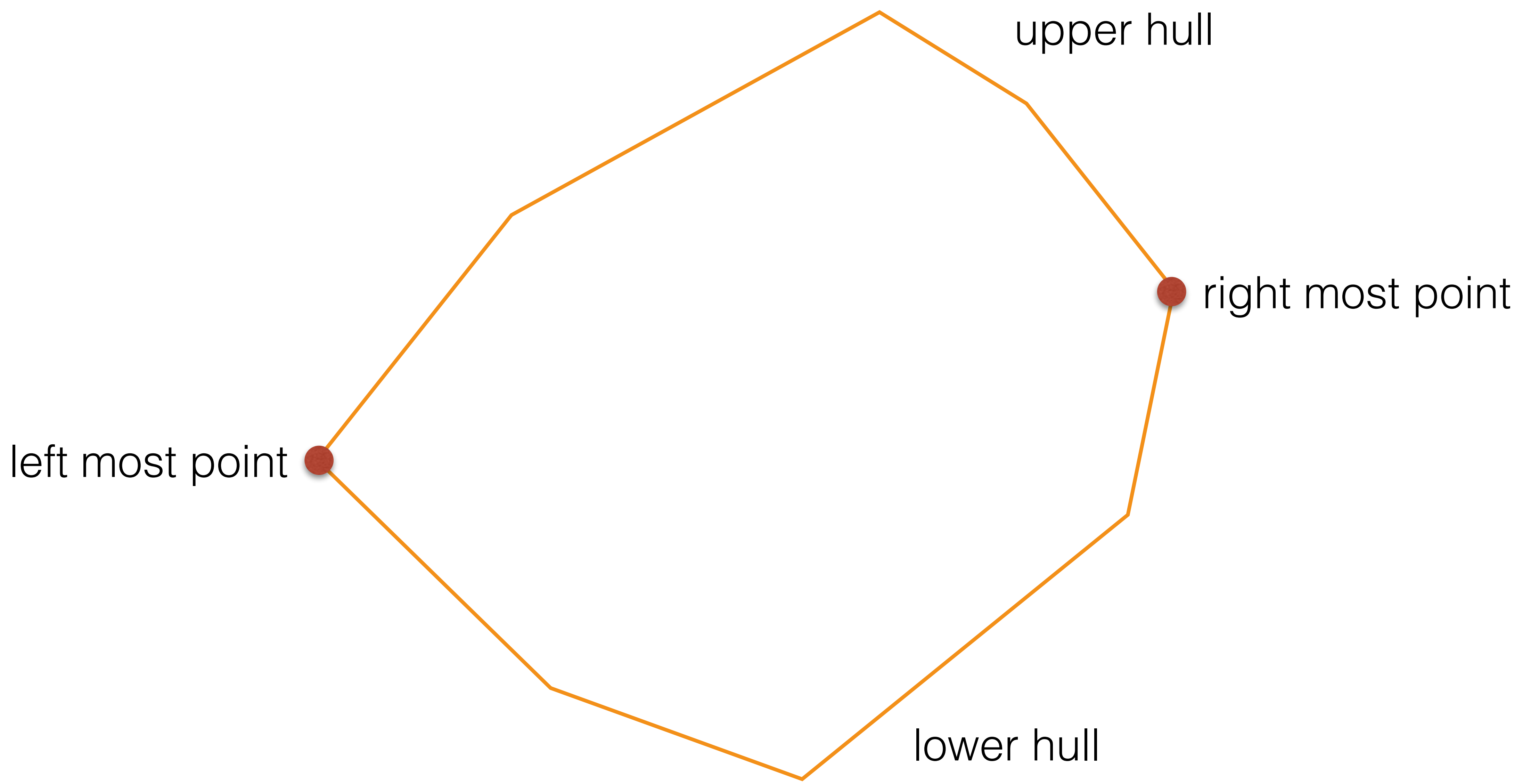
- Simulate Gift Wrapping on an arbitrary (small) set of points
    - consider how it works in degenerate cases
- Analysis: Running time? Express function of n and k, where k is the output size (number of points on the convex hull)
    - How small/large can k be for a set of n points?
    - Show examples that trigger best/worst cases
    - Based on this, discuss when gift-wrapping is a good choice

Summary

- **Gift wrapping algorithm**
  - Runs in O(kn) time, where k is the size of the CH(P)
  - Efficient if k is small:
    - For k = O(1), it takes O(n)
  - Not efficient if k is large:
    - For k = O(n), gift wrapping takes $O(n^2)$
  - Faster algorithms are known
  - Gift wrapping extends easily to 3D and for many years was the primary algorithm for 3D

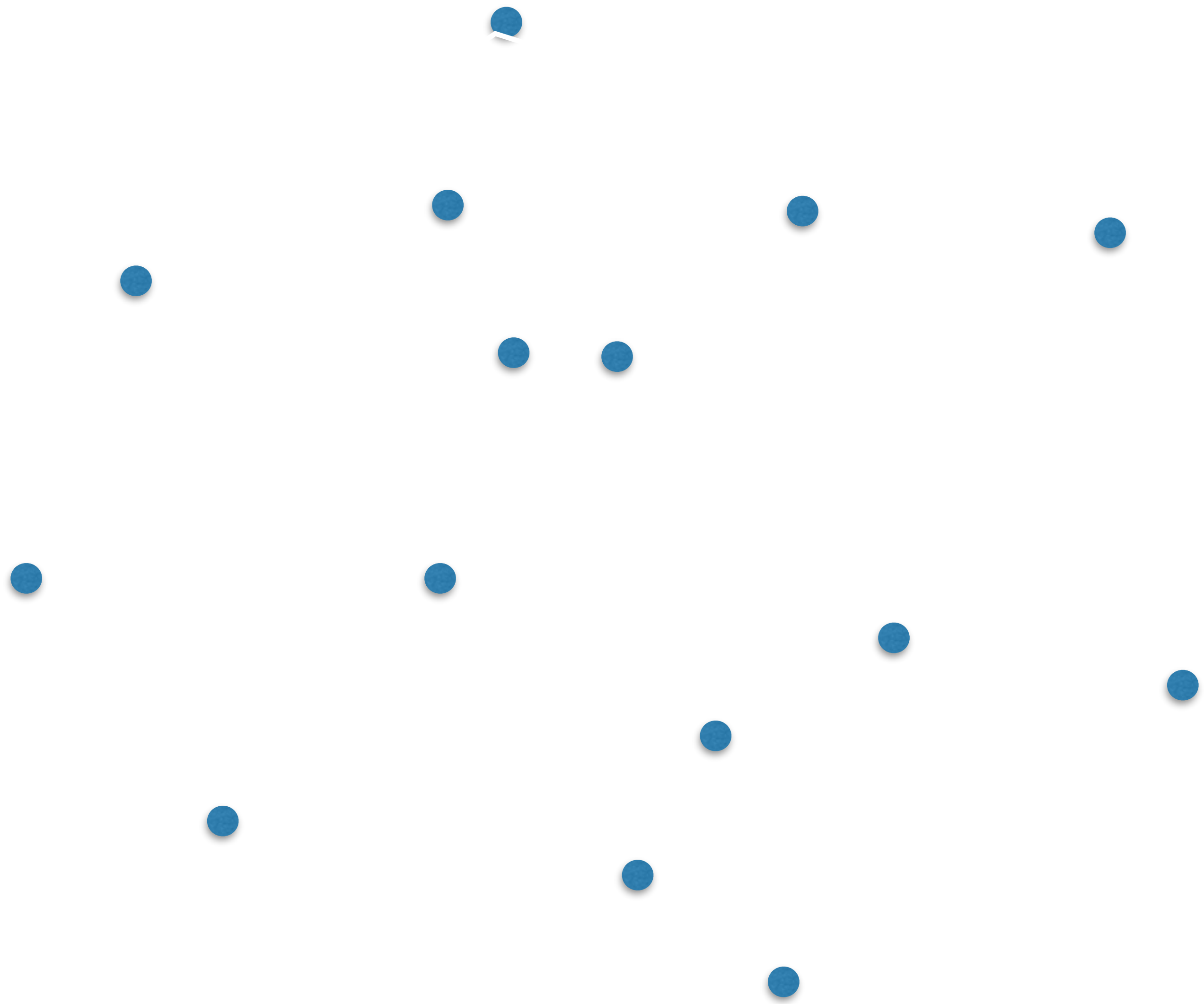
Quickhull

Convex polygons: Properties



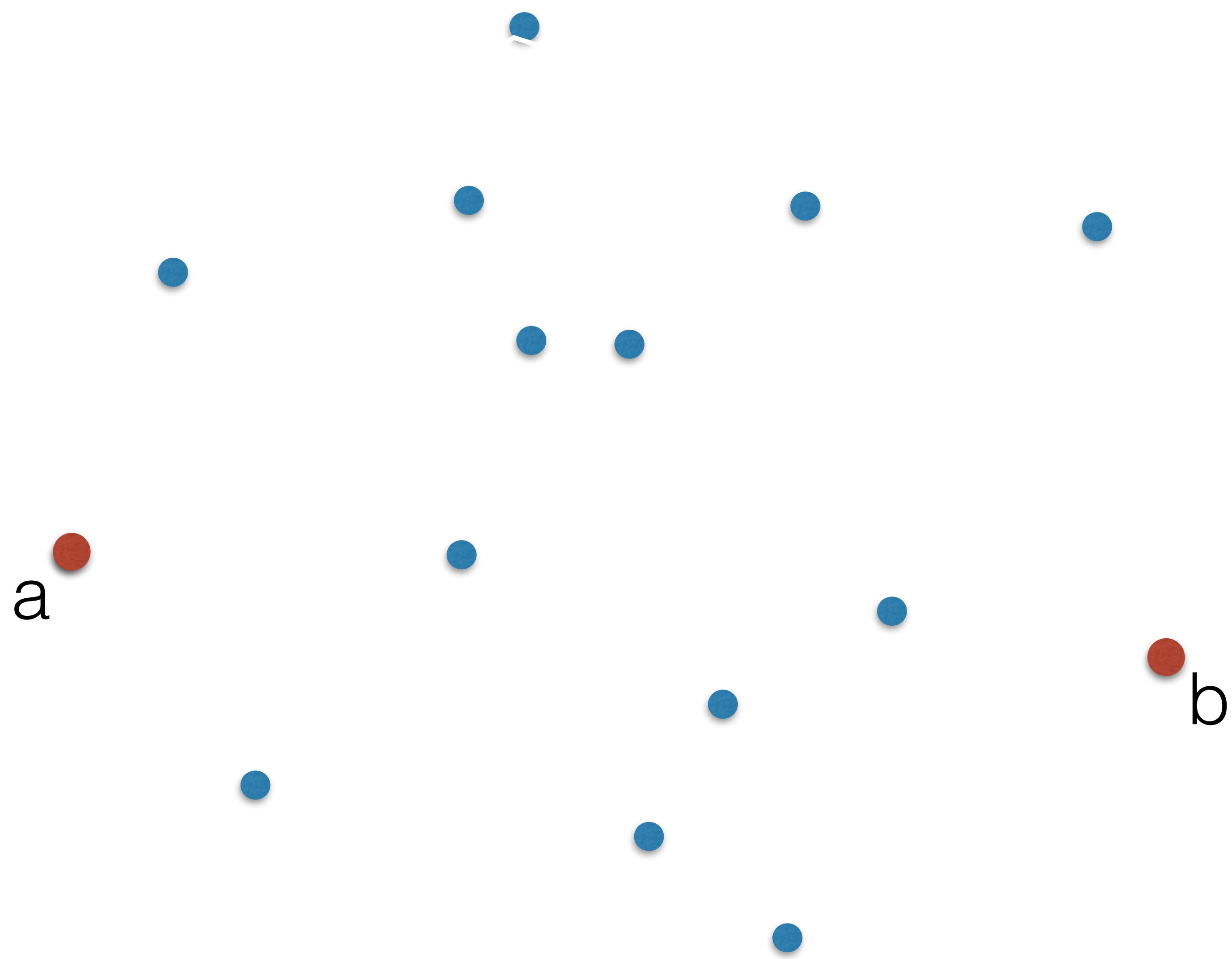
Quickhull (late 1970s)

- Similar to Quicksort (in some way)



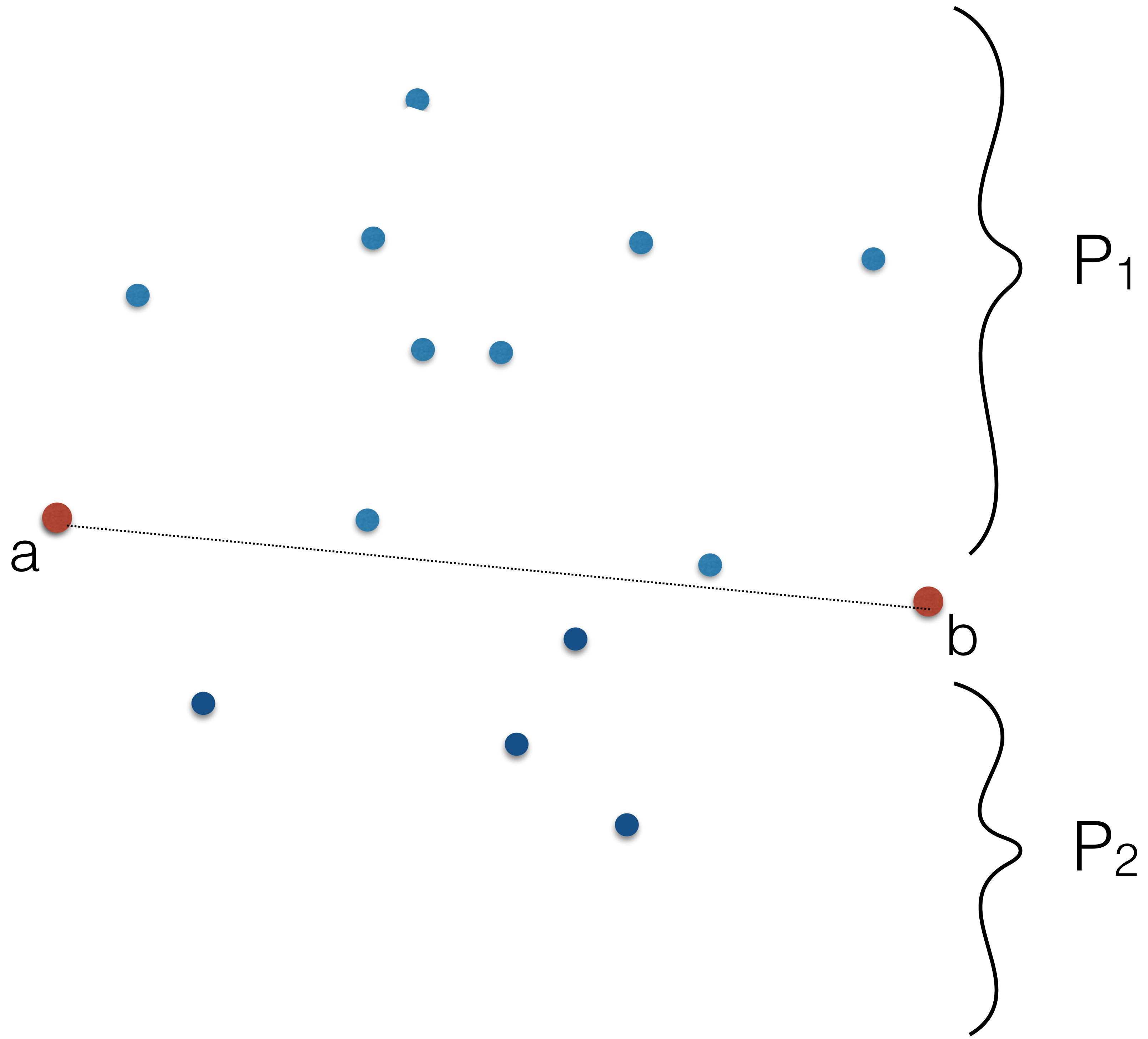
Quickhull (late 1970s)

- Idea: start with 2 extreme points



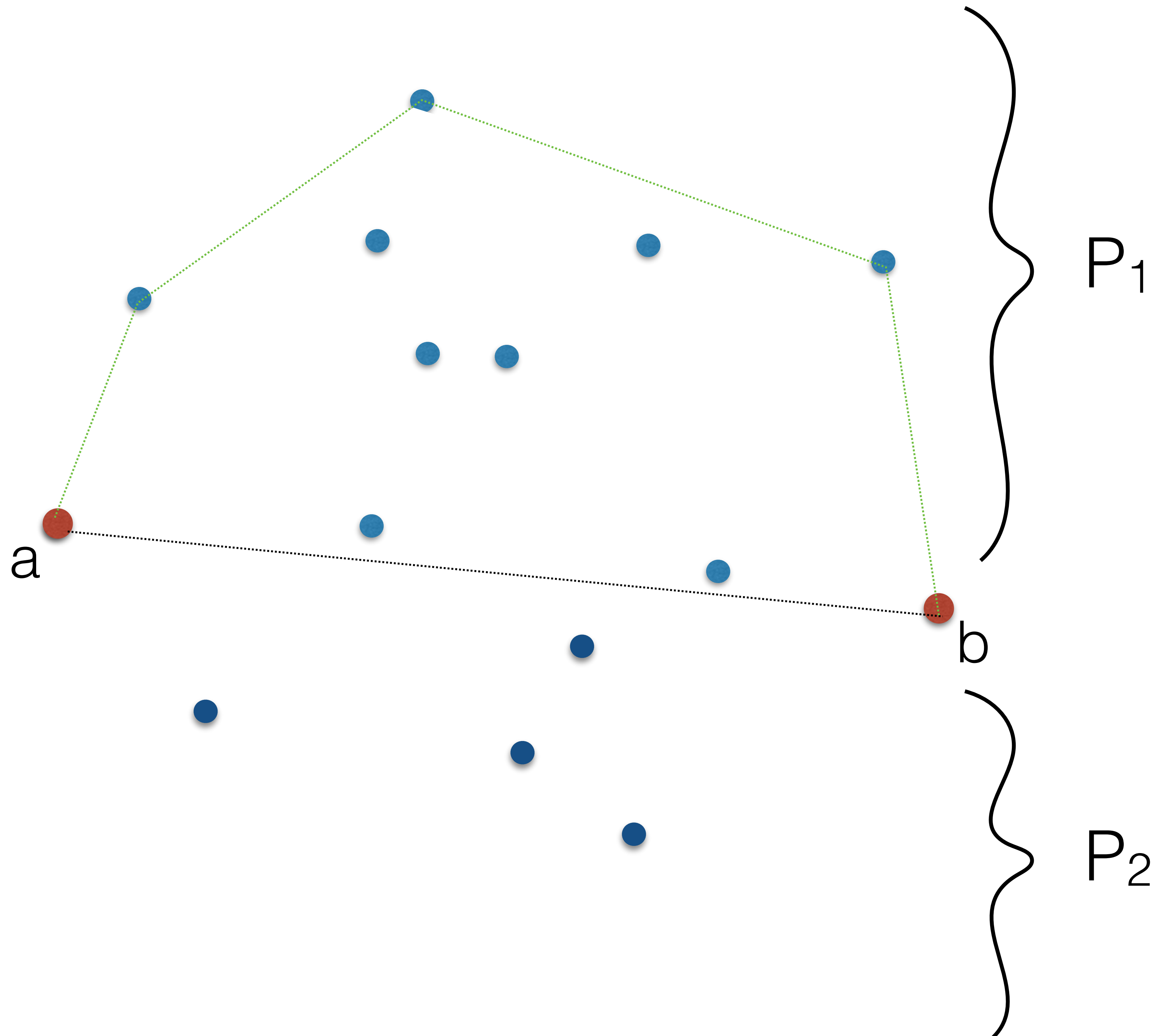
Quickhull (late 1970s)

- CH = upper hull (CH of P_1) + lower hull (CH of P_2)



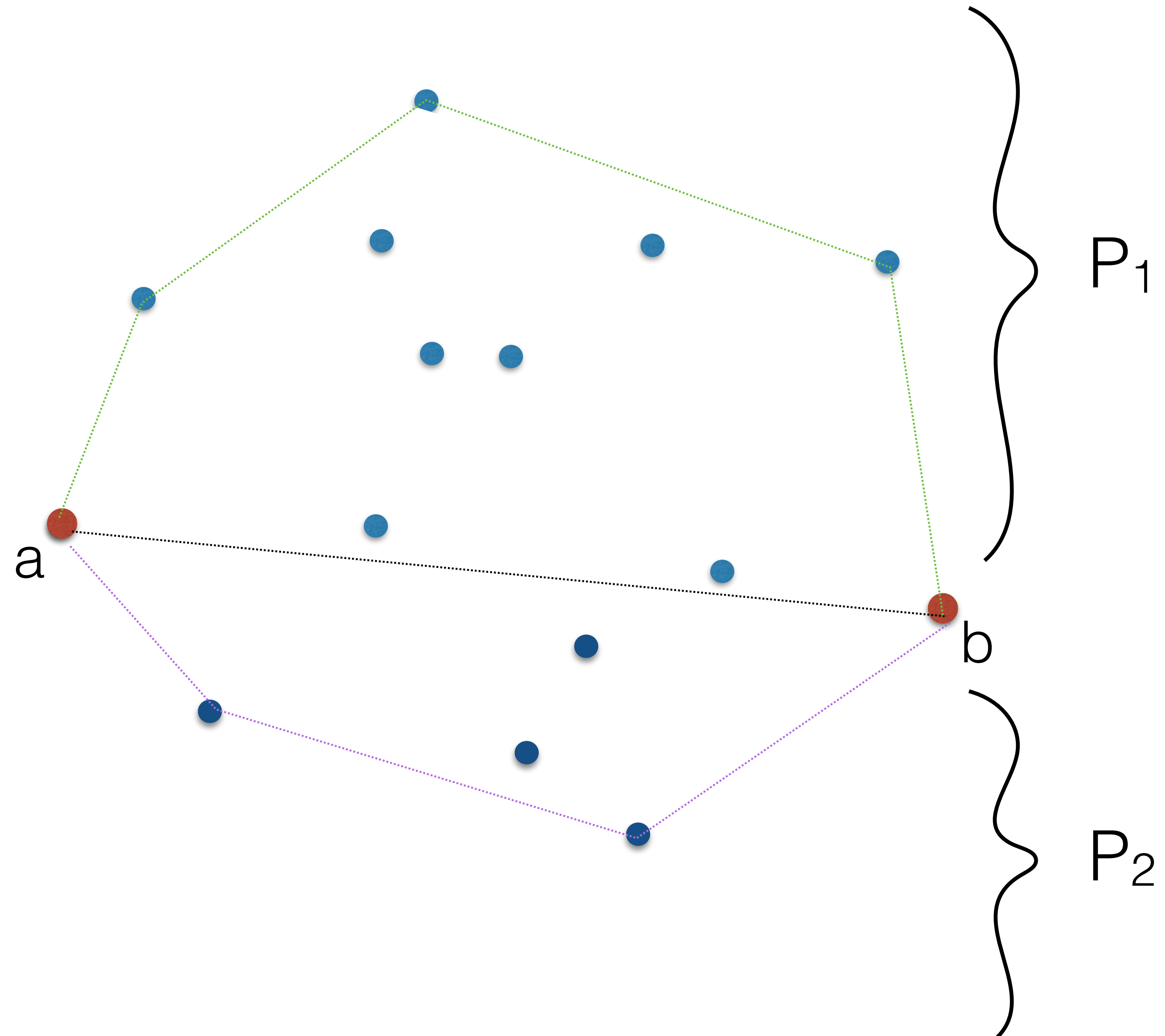
Quickhull (late 1970s)

- CH = upper hull (CH of P_1) + lower hull (CH of P_2)



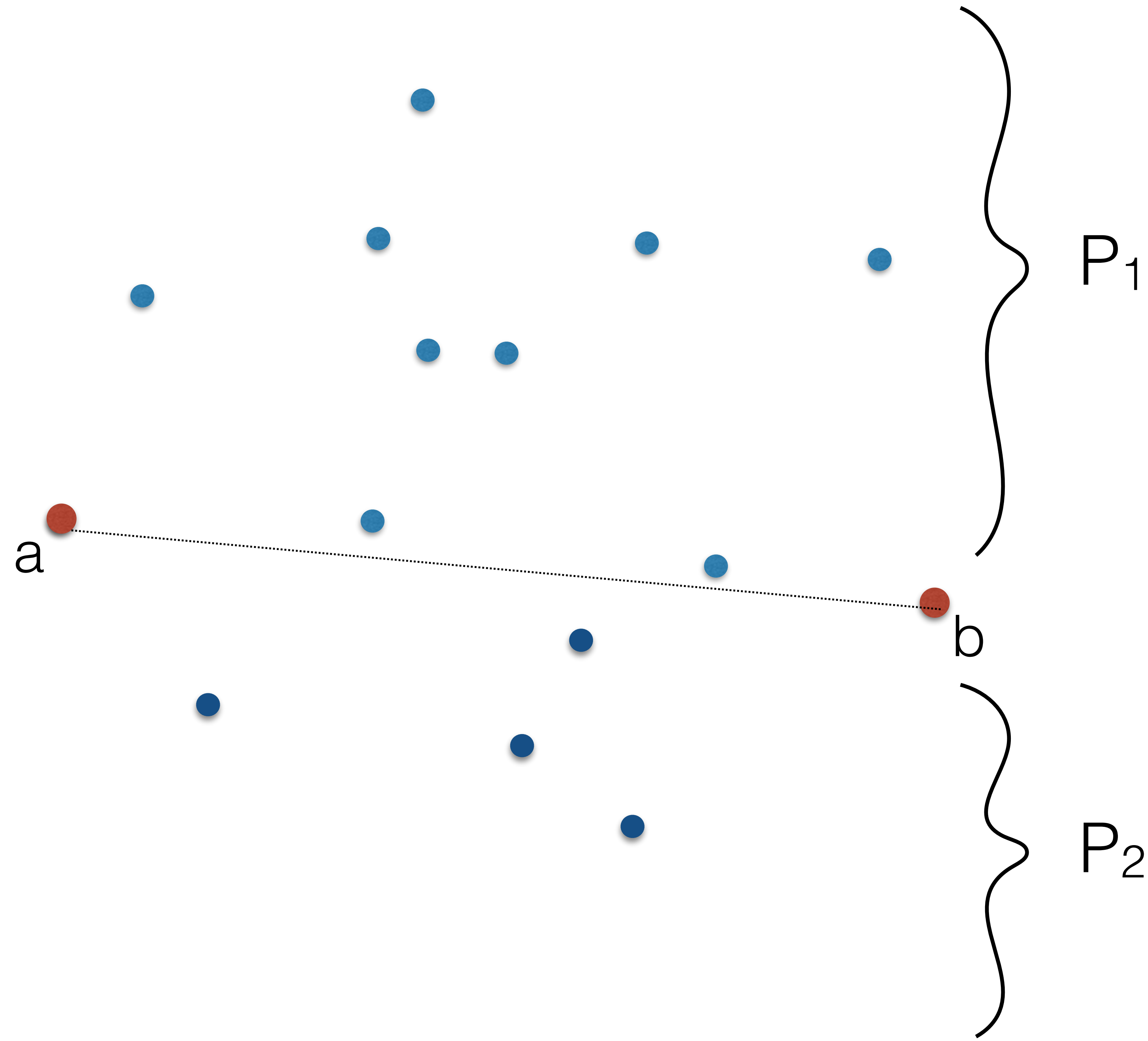
Quickhull (late 1970s)

- CH = upper hull (CH of P_1) + lower hull (CH of P_2)



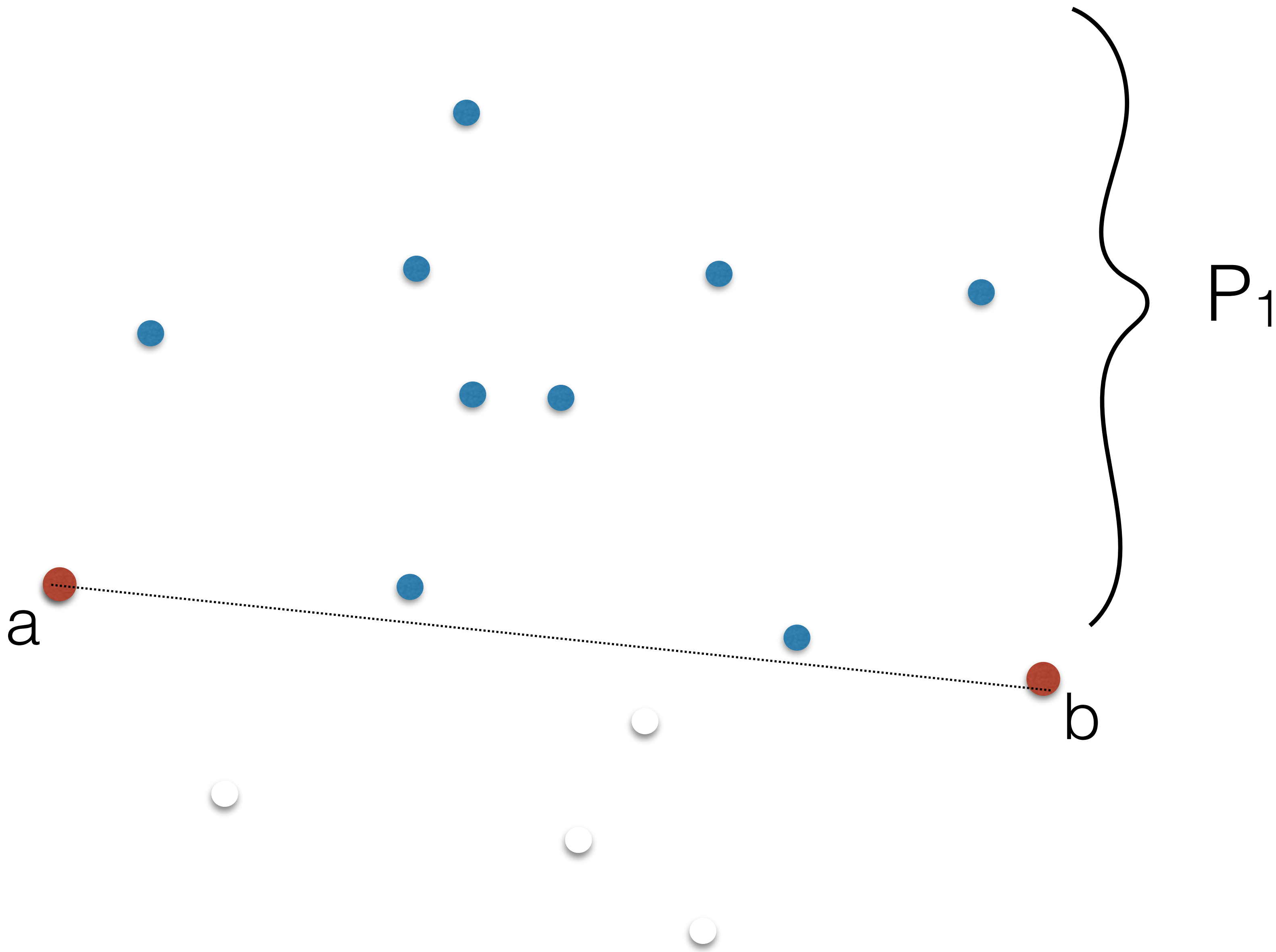
Quickhull (late 1970s)

- We'll find the $CH(P_1)$ and $CH(P_2)$ separately



Quickhull (late 1970s)

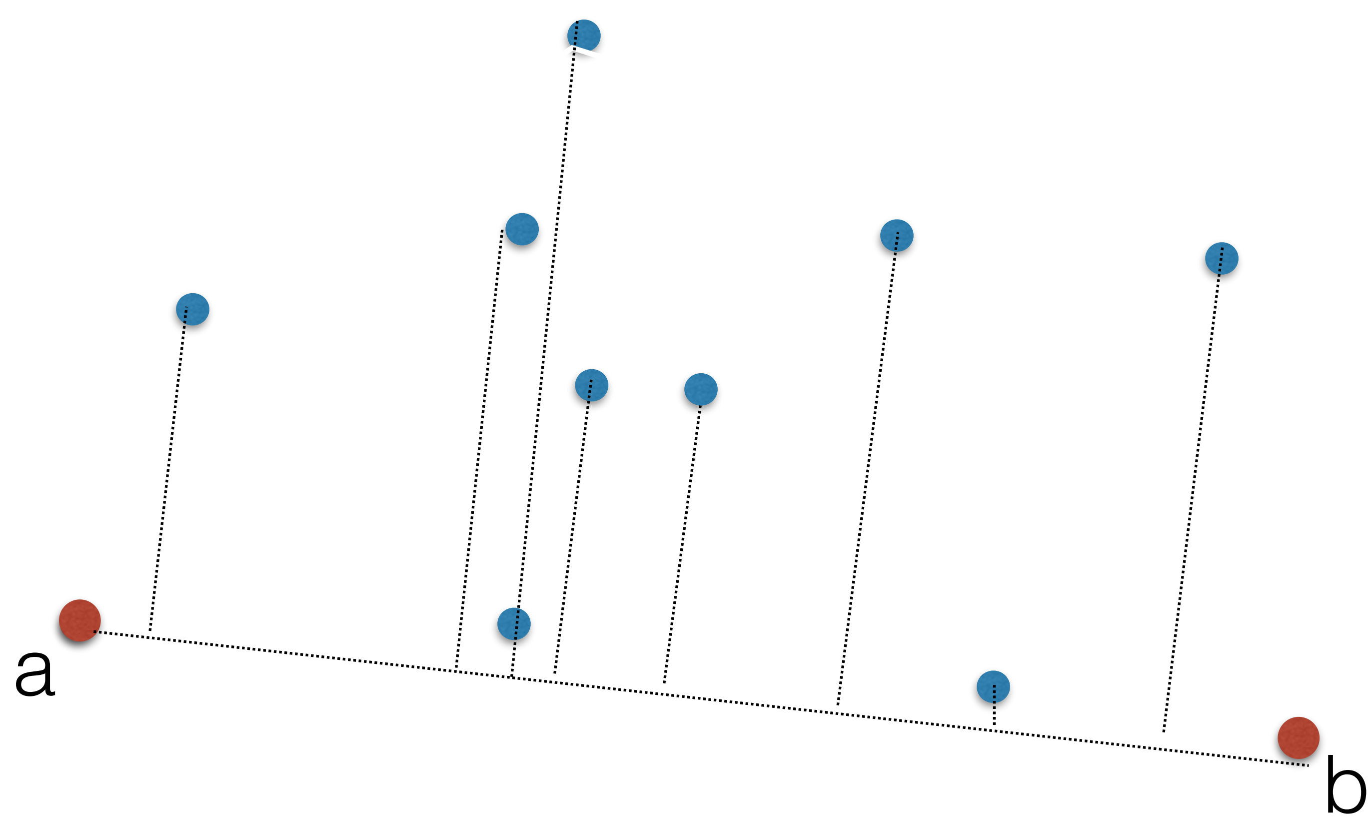
- First let's focus on P1



Quickhull (late 1970s)

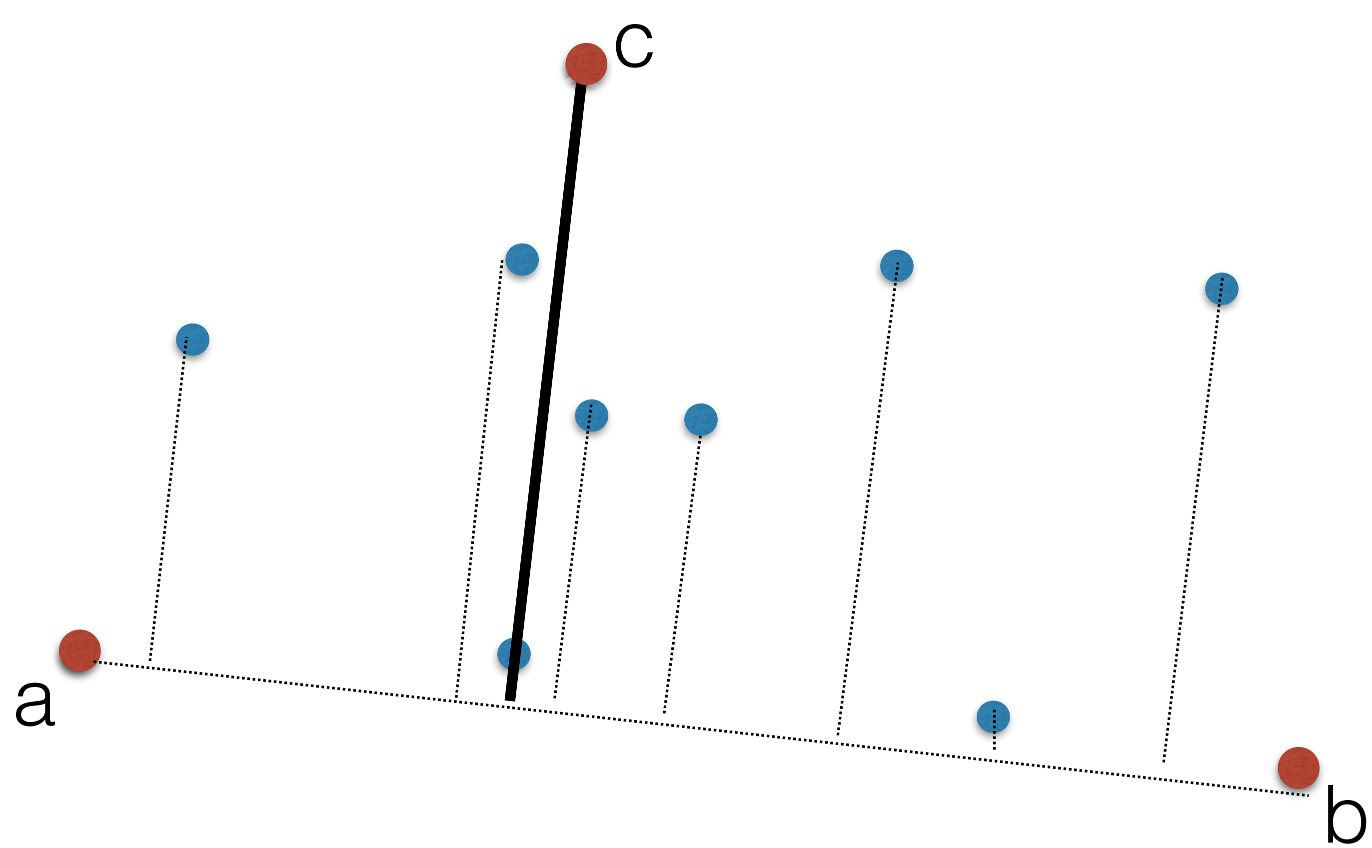
let's ignore collinear points for now

- For all points p in P1: compute dist(p, ab)



Quickhull (late 1970s)

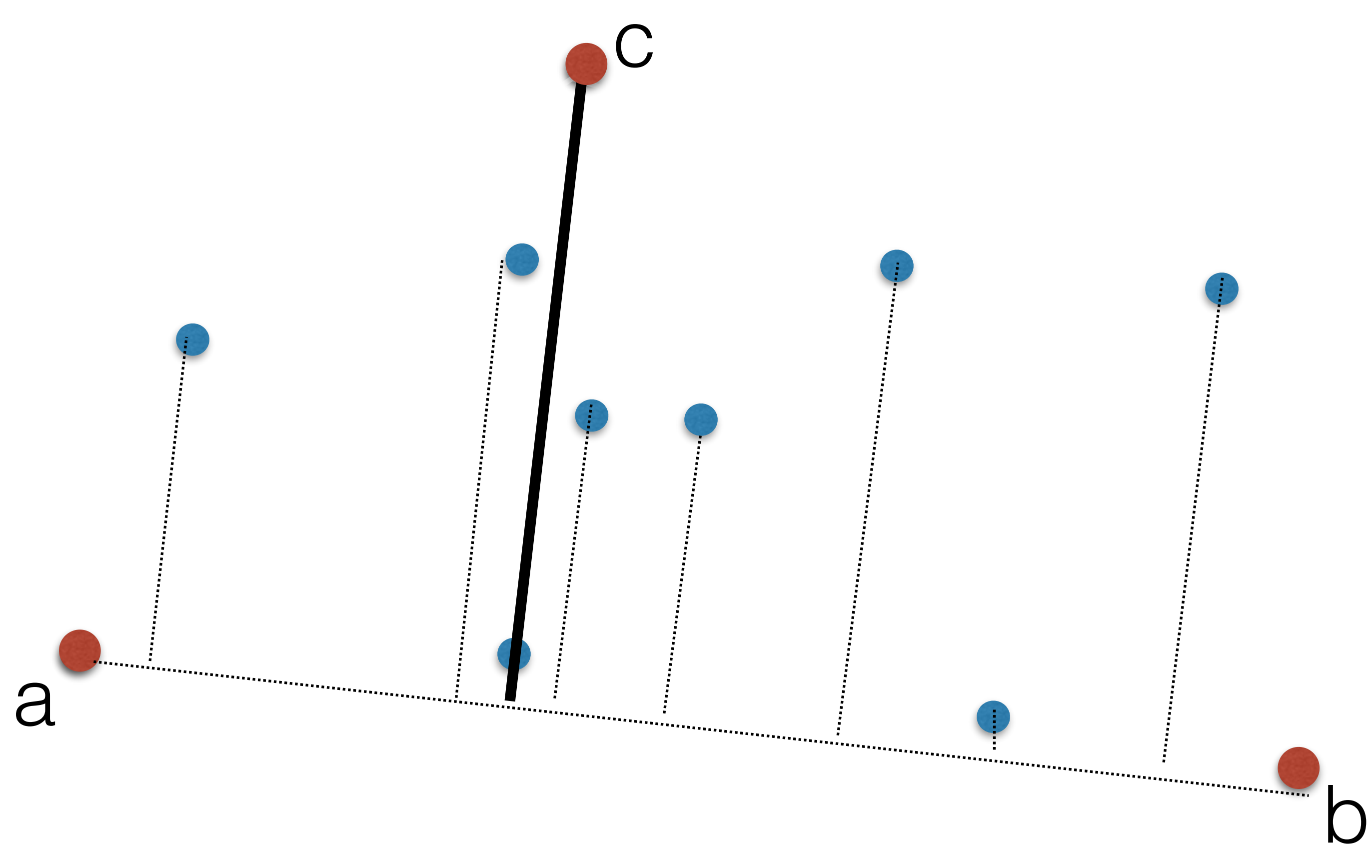
- Find the point c with largest distance (i.e. furthest away from ab)



Quickhull (late 1970s)

let's ignore collinear points for now

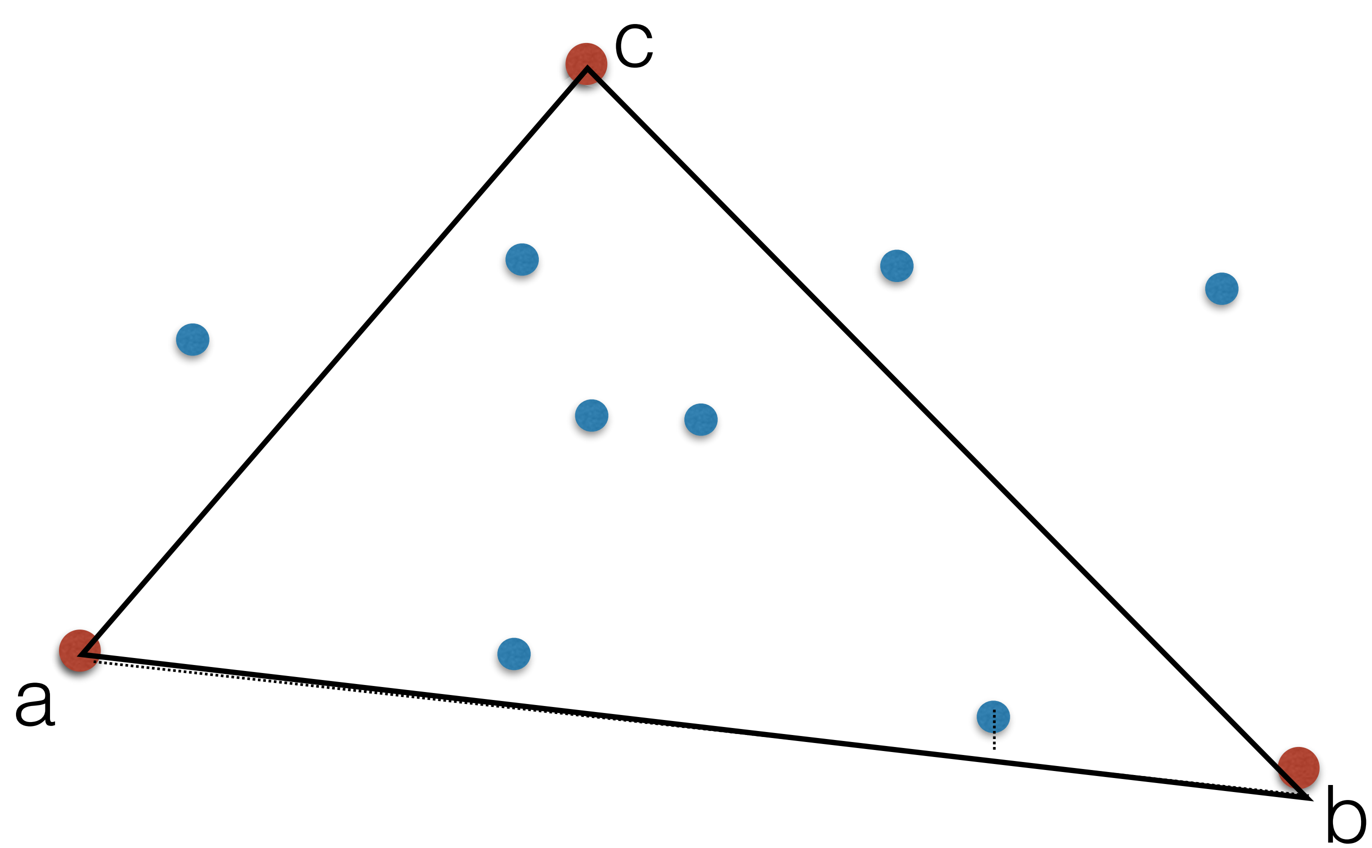
- Find the point c with largest distance (i.e. furthest away from ab)



- Claim: c must be an extreme point (and thus on the CH of P1)
- Why?

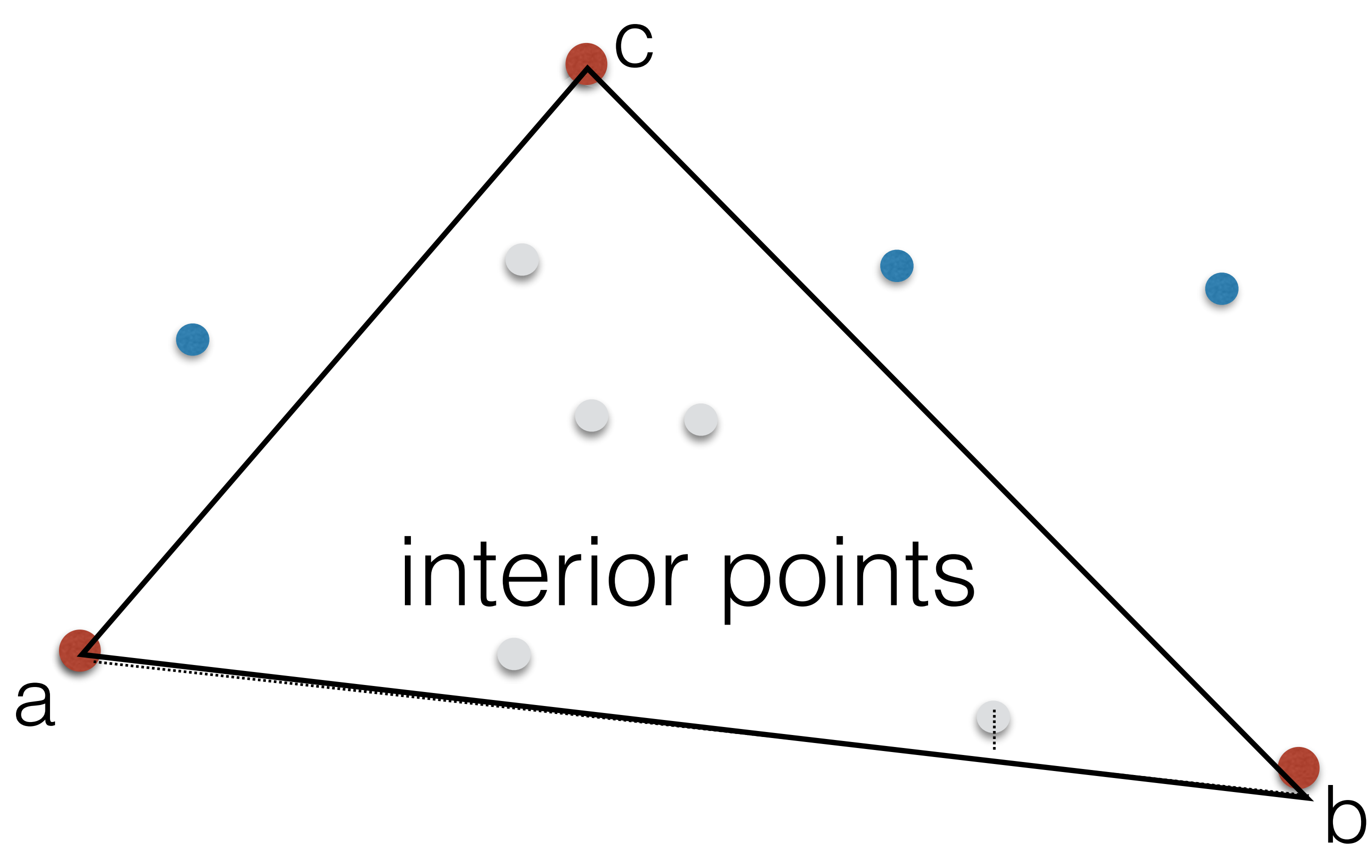
Quickhull (late 1970s)

- Discard all points inside triangle abc



Quickhull (late 1970s)

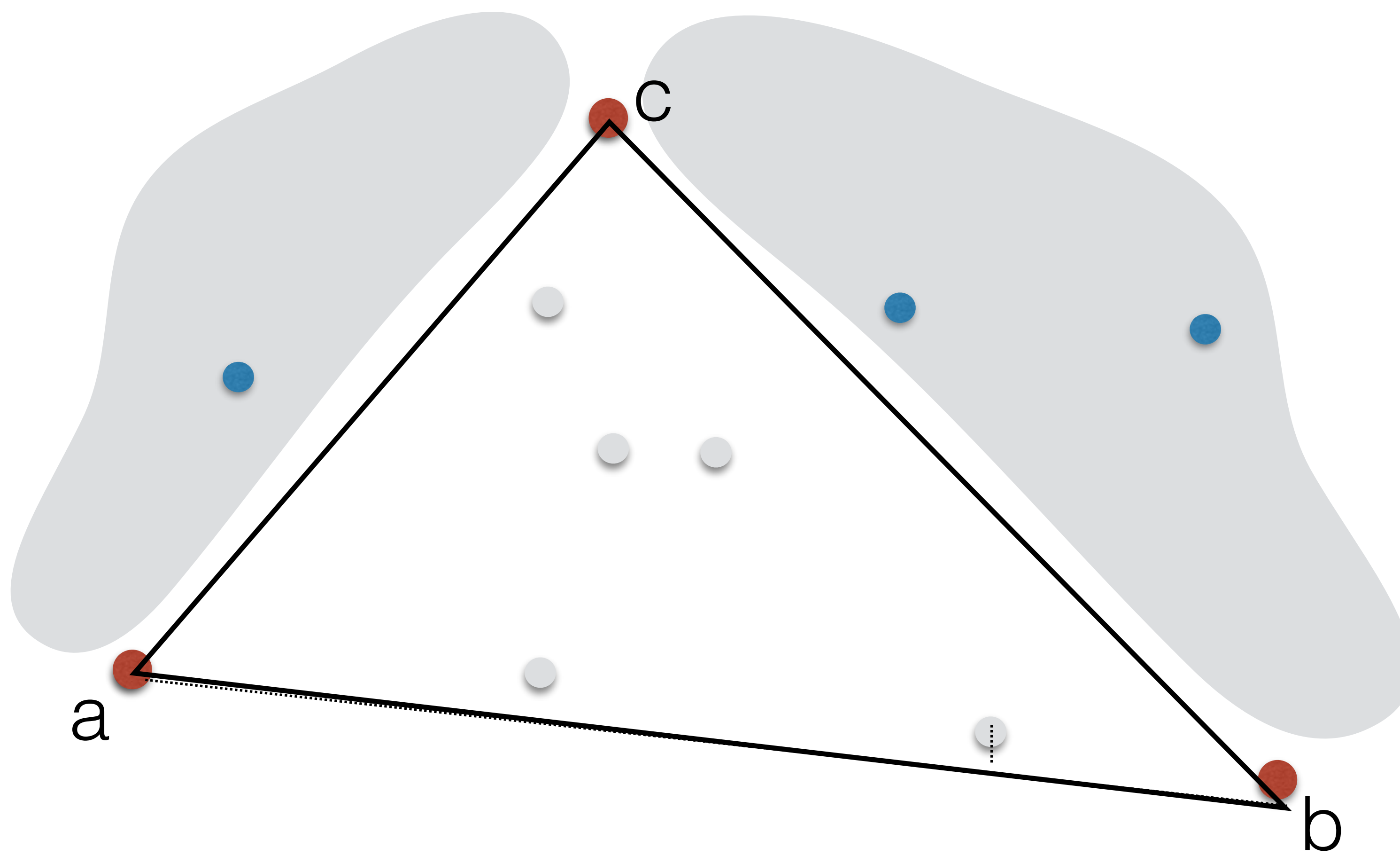
- Discard all points inside triangle abc



Quickhull (late 1970s)

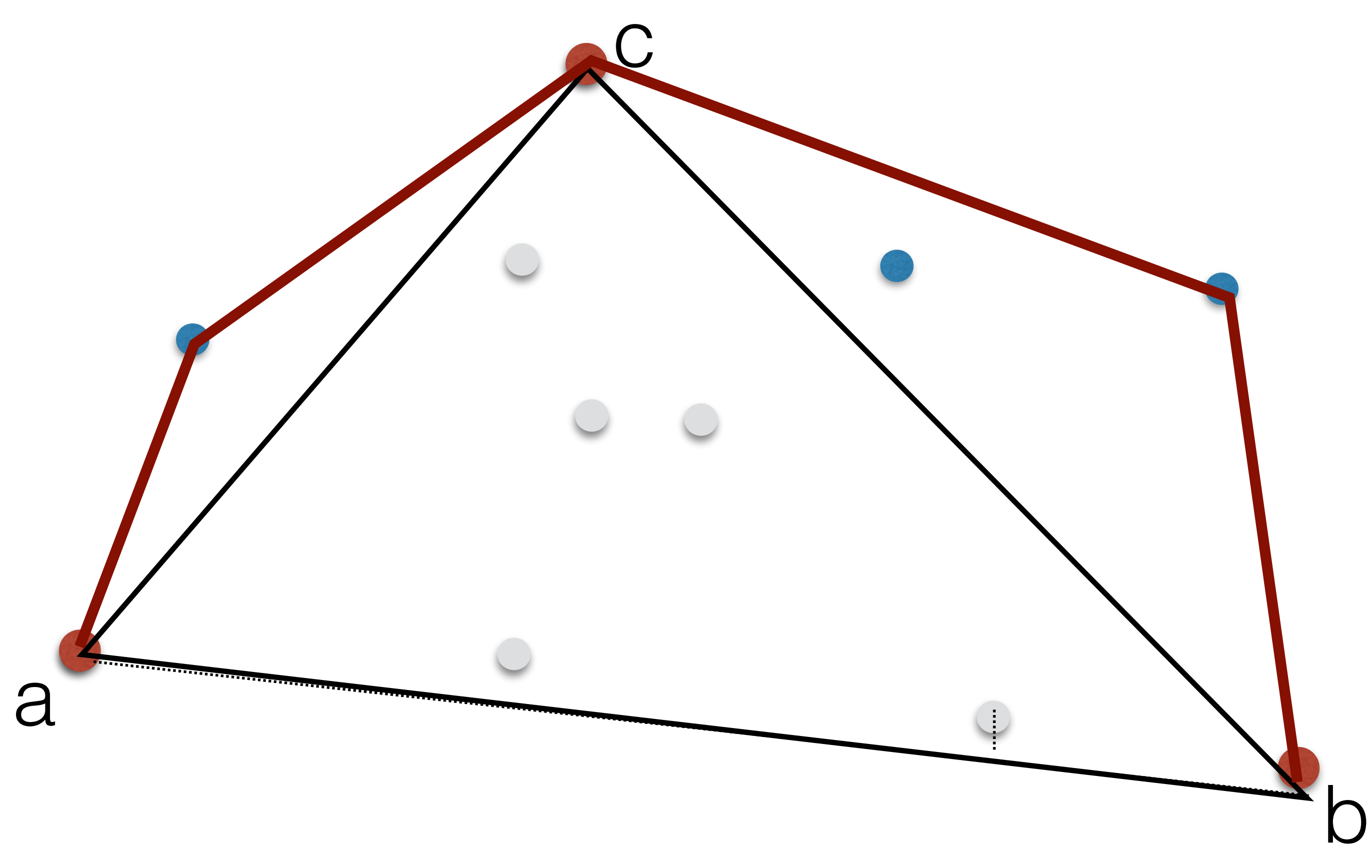
let's ignore collinear points for now

- **Recurse** on the points left of ac and right of bc



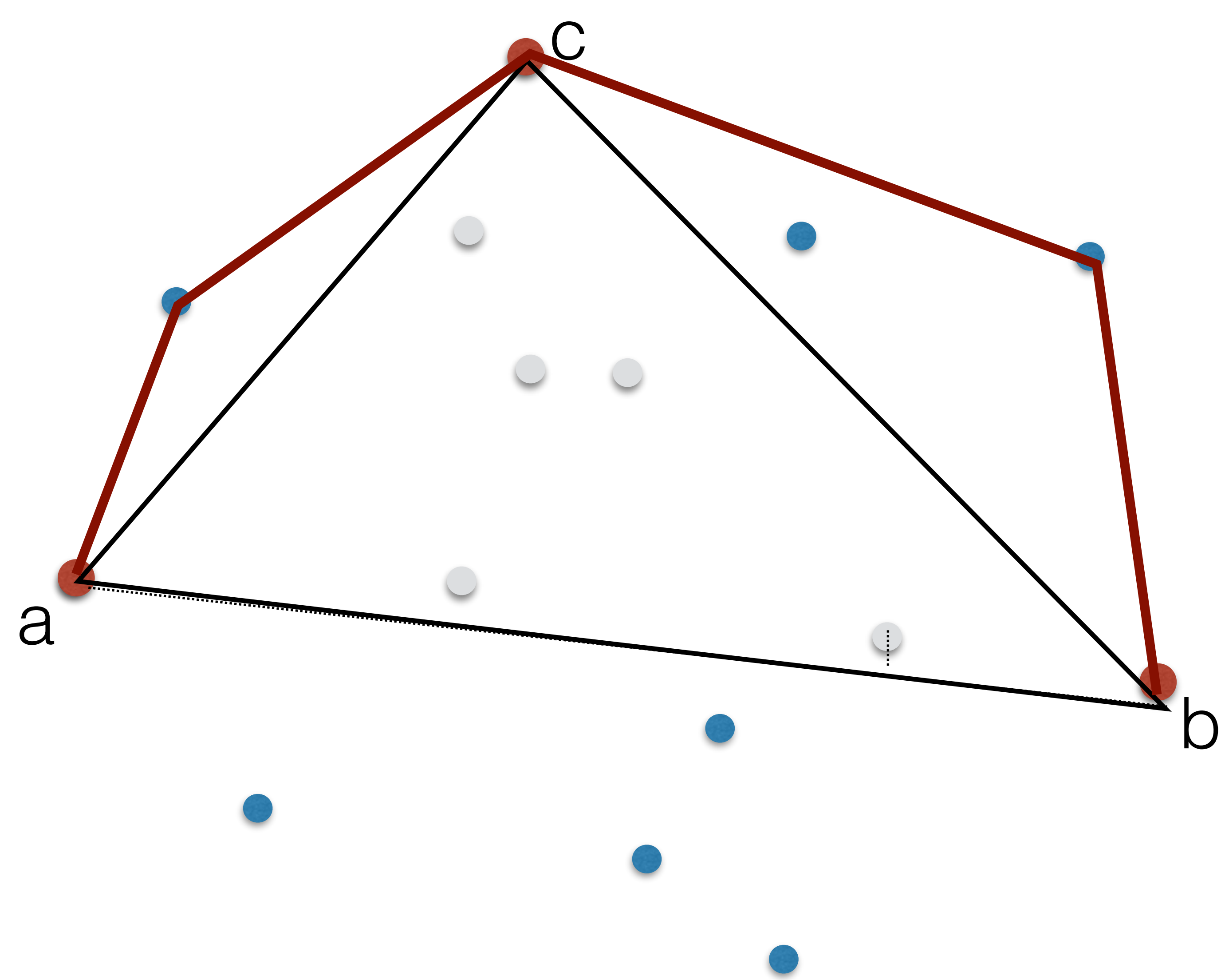
Quickhull (late 1970s)

- **Recurse** on the points left of ac and right of bc

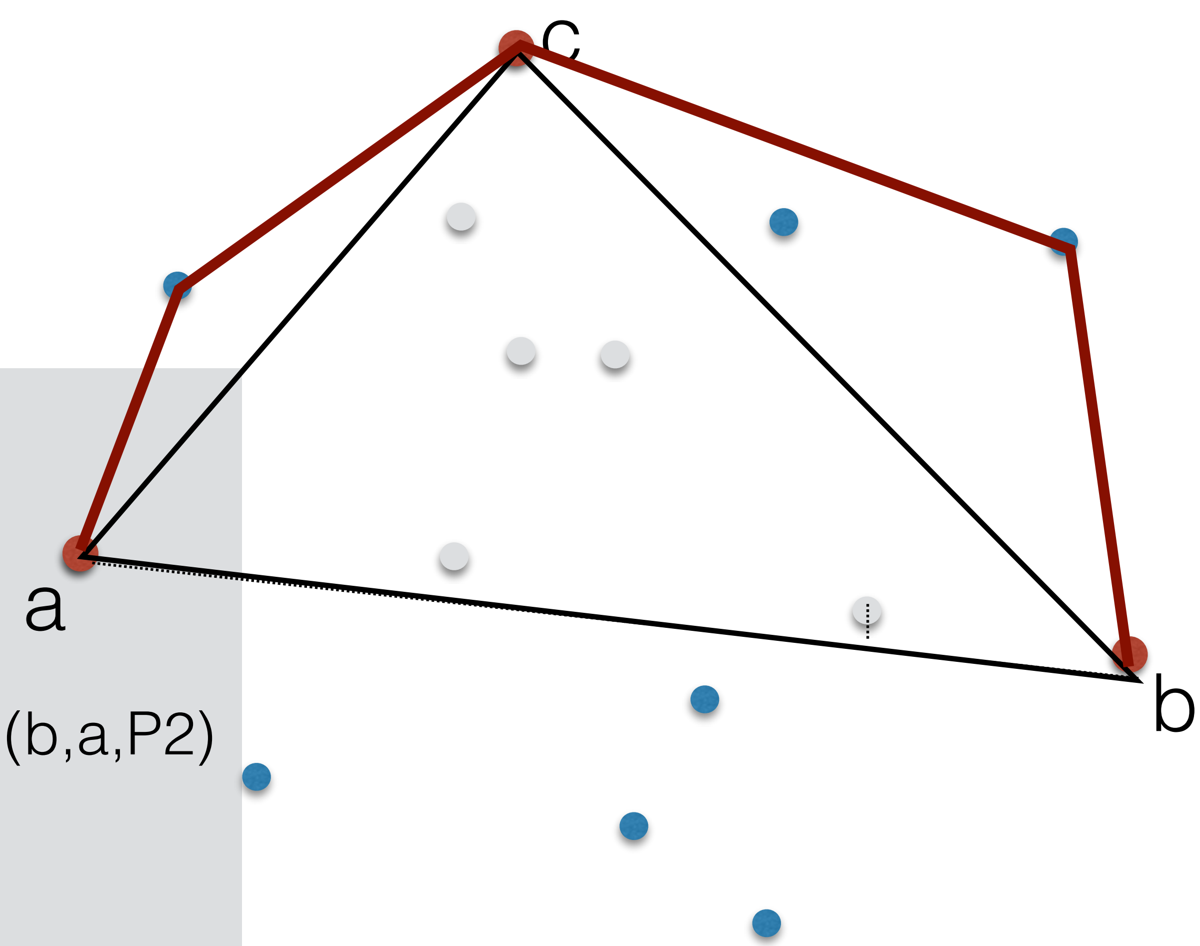


Quickhull (late 1970s)

- Compute CH of P_2 similarly



Quickhull (late 1970s)



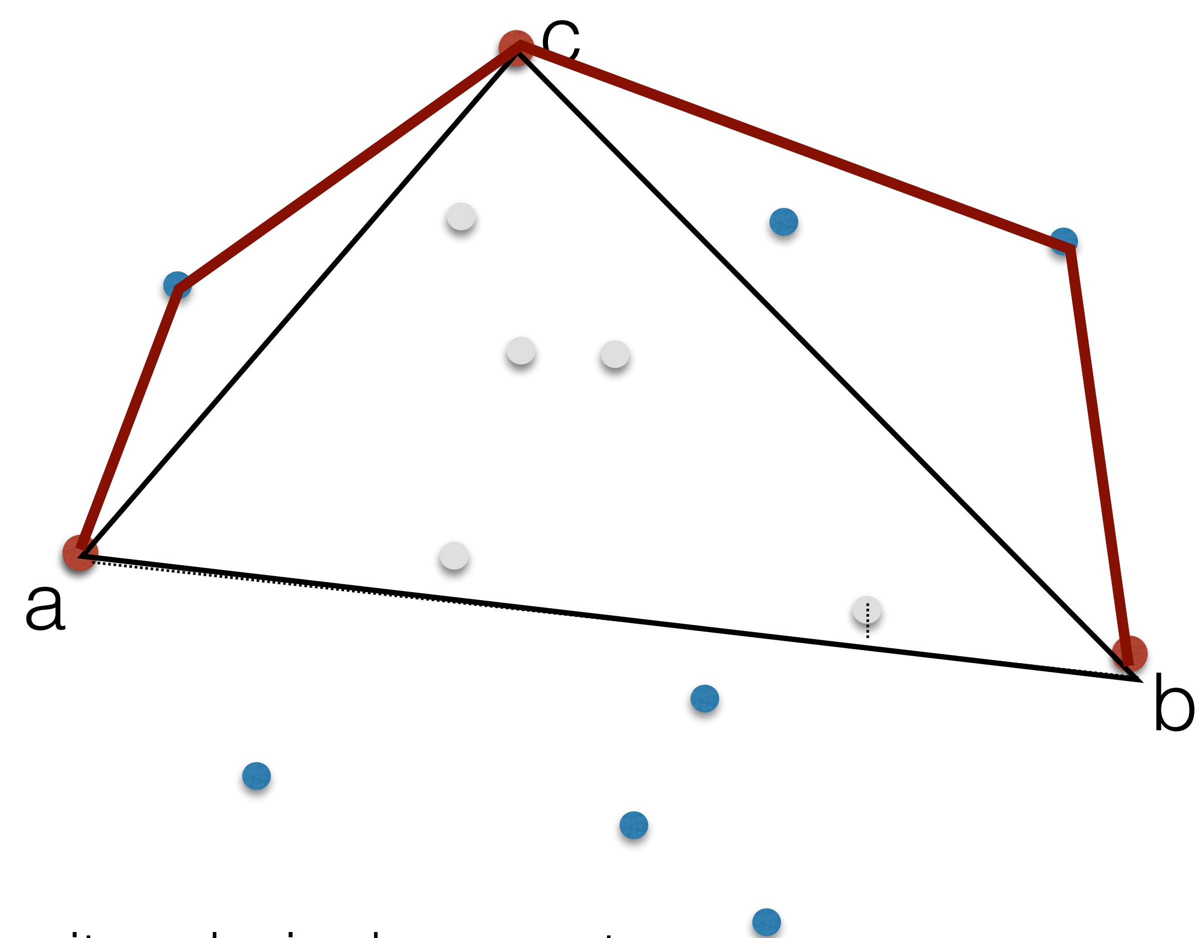
- **Quickhull (P)**
  - find a, b
  - partition P into P1, P2
  - return a + Quickhull(a,b, P1) + b + Quickhull(b,a,P2)

- **Quickhull(a,b,P)**

  //invariant: P is a set of points all on the left of ab

  - if P empty => return emptyset
  - for each point p in P: compute its distance to ab
  - let c = point with max distance
  - let P1 = points to the left of ac
  - let P2 = points to the left of cb
  - return Quickhull(a,c,P1) + c + Quickhull(c,b,P2)

Quickhull : Classwork



- Simulate Quickhull on a set of points and think how it works in degenerate cases

- Analysis:
    - Write a recurrence relation for its running time
    - What/when is the worst case running time ?
    - What/when is the best case running time ?

- Argue that Quickhull's average complexity is O(n) on points that are uniformly distributed.

Summary

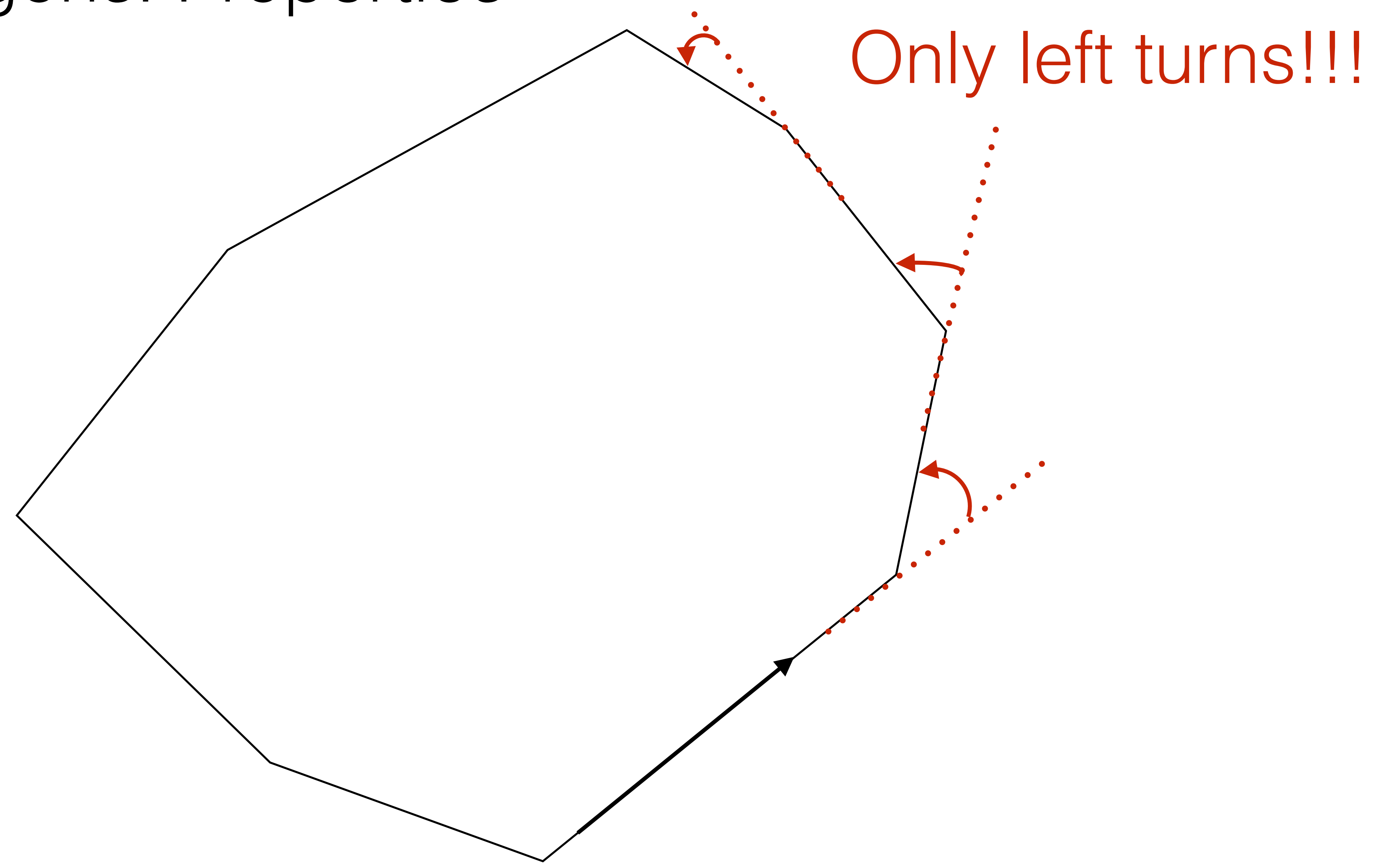
- Convex Hull algorithms so far
  - Brute force: $O(n^3)$
  - Gift wrapping: $O(kn)$
    - output-size sensitive: $O(n)$ best case, $O(n^2)$ worst case
    - ✦ by Chand and Kapur [1970]. Extends to 3D and to arbitrary dimensions; for many years was the primary algorithms for higher dimensions
  - Quickhull: $O(n^2)$
  - Next
    - Graham scan
    - lower bound
    - other approaches: incremental, divide-and-conquer

Graham scan

Graham scan (late 1960s)

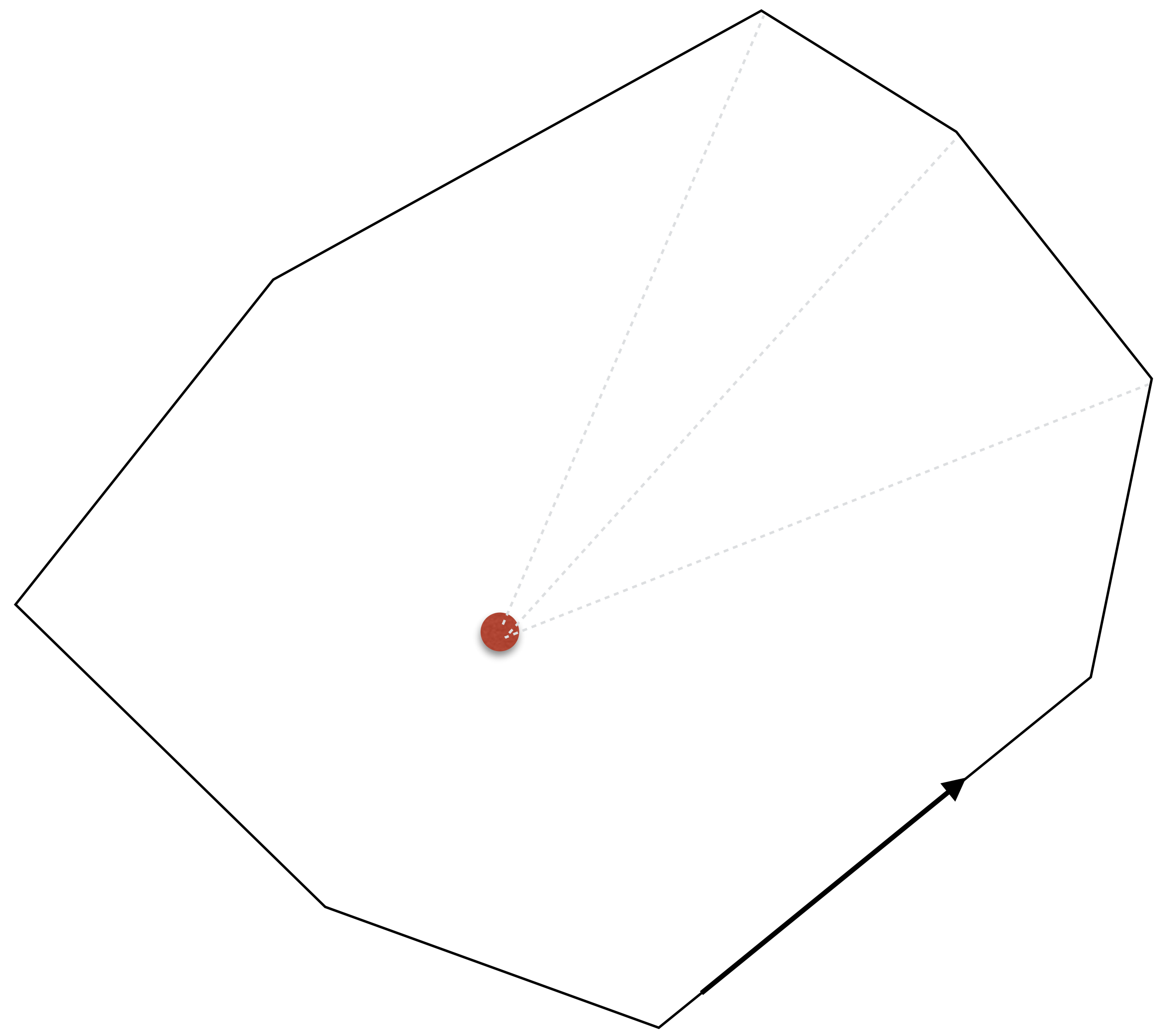
- In late 60s an application at Bell Labs required the hull of 10,000 points, for which a quadratic algorithm was too slow
- Graham developed an algorithm which runs in O(n lg n)
    - It runs in one sort plus a linear pass!!
    - Simple, intuitive, elegant and practical
    - You’ll love it

Convex polygons: Properties



Walk ccw along the boundary of a convex polygon

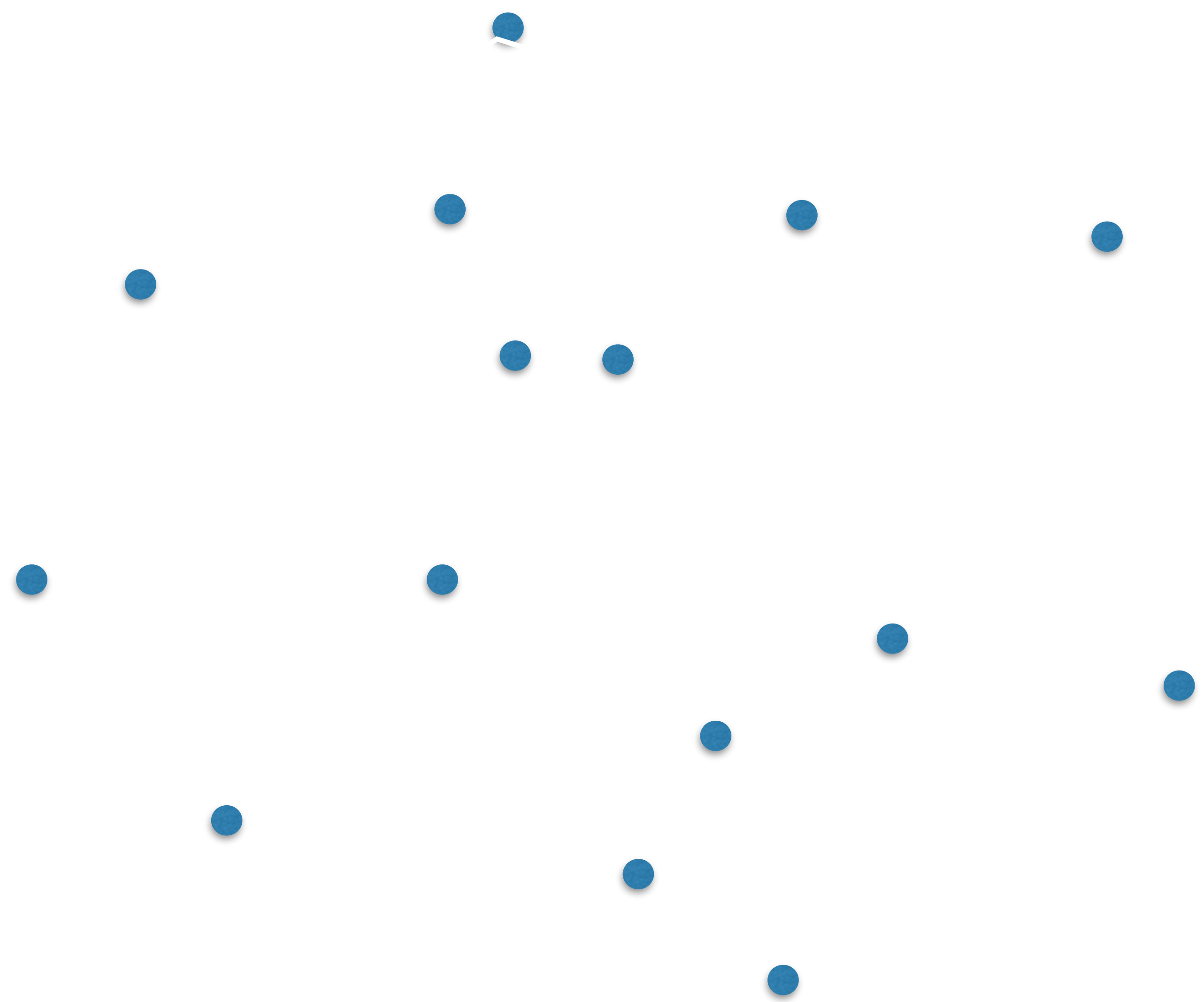
Convex polygons: Properties



Walk ccw along the boundary of a convex polygon

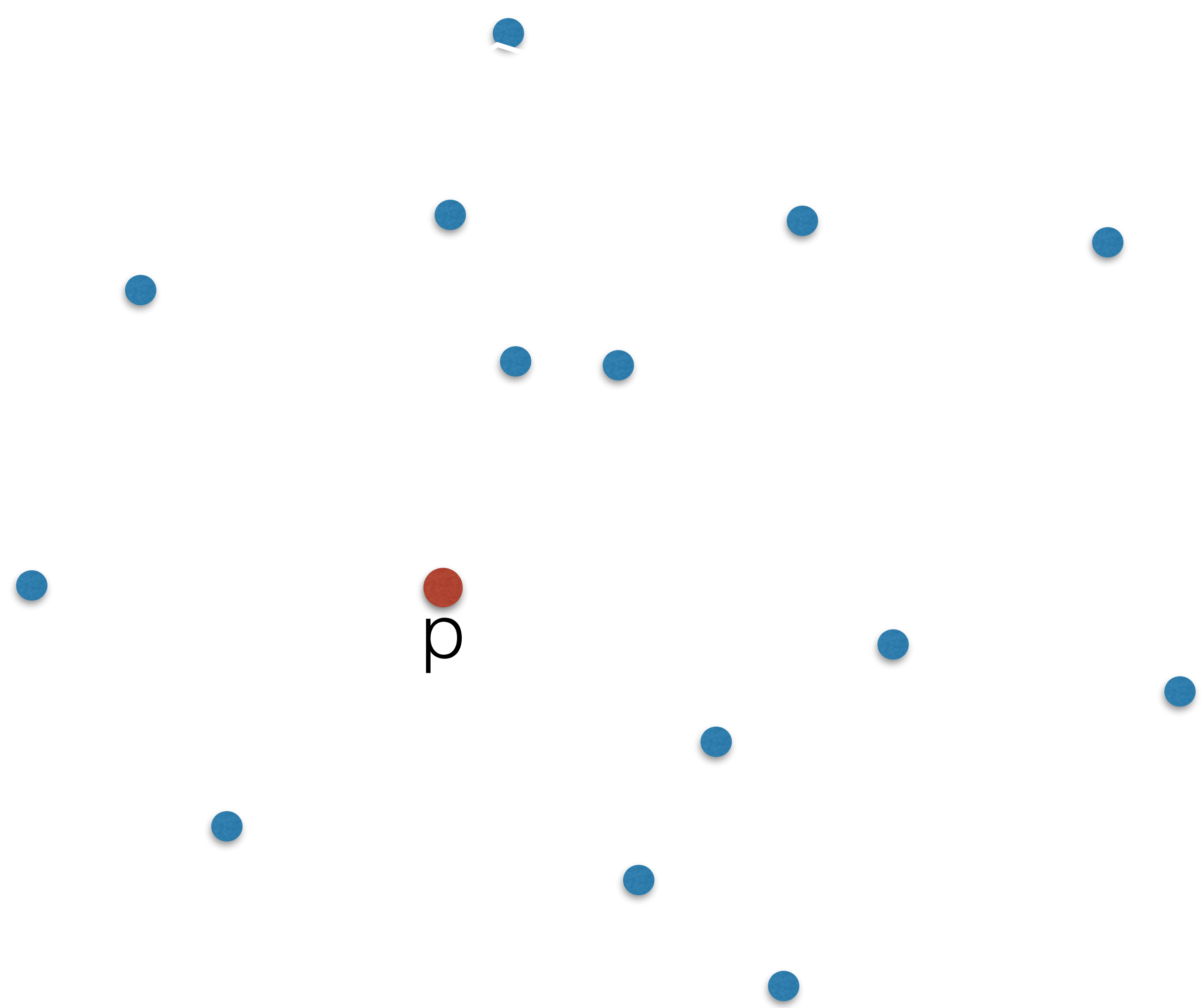
For any point p inside, the points on the boundary are in radial order around p

Graham scan (late 1960s)



Graham scan (late 1960s)

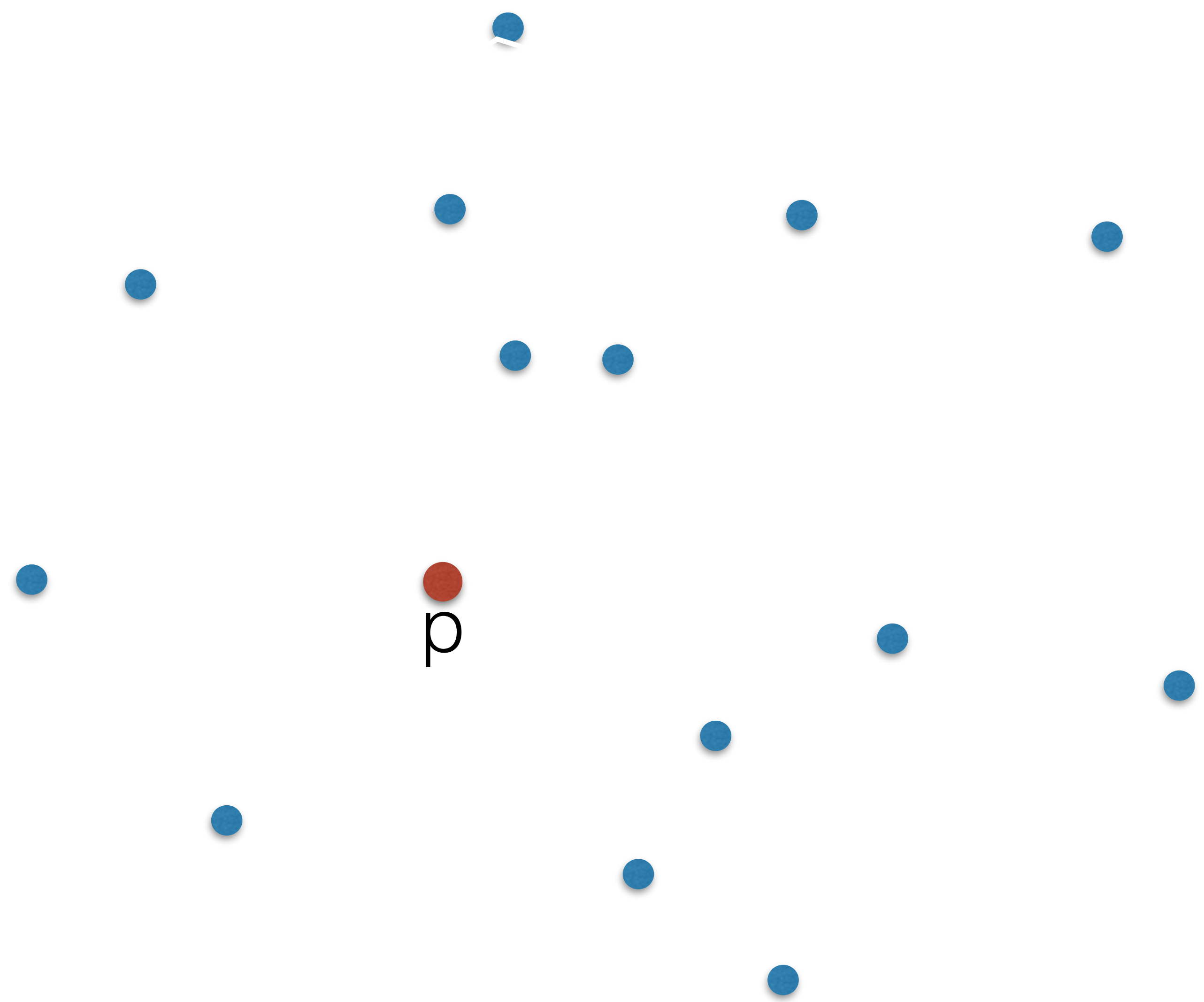
- Idea: start from a point p interior to the hull <———- we'll think about how to get it later



Graham scan (late 1960s)

- Idea: start from a point p interior to the hull <——– we'll think about how to get it later

  order all points by their ccw angle wrt p



Graham scan (late 1960s)

- Idea: start from a point p interior to the hull

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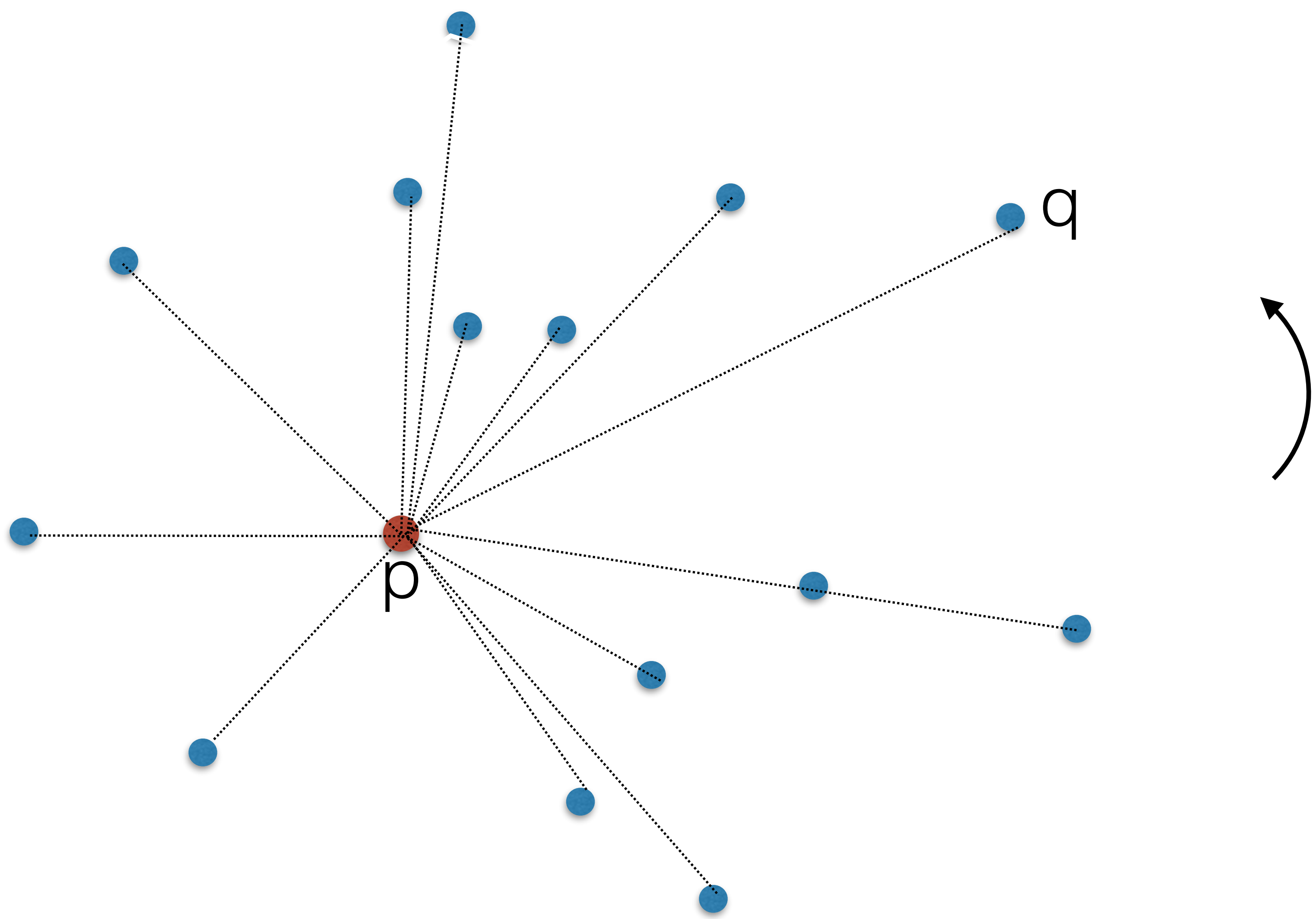

Graham scan (late 1960s)

- Idea: start from a point p interior to the hull

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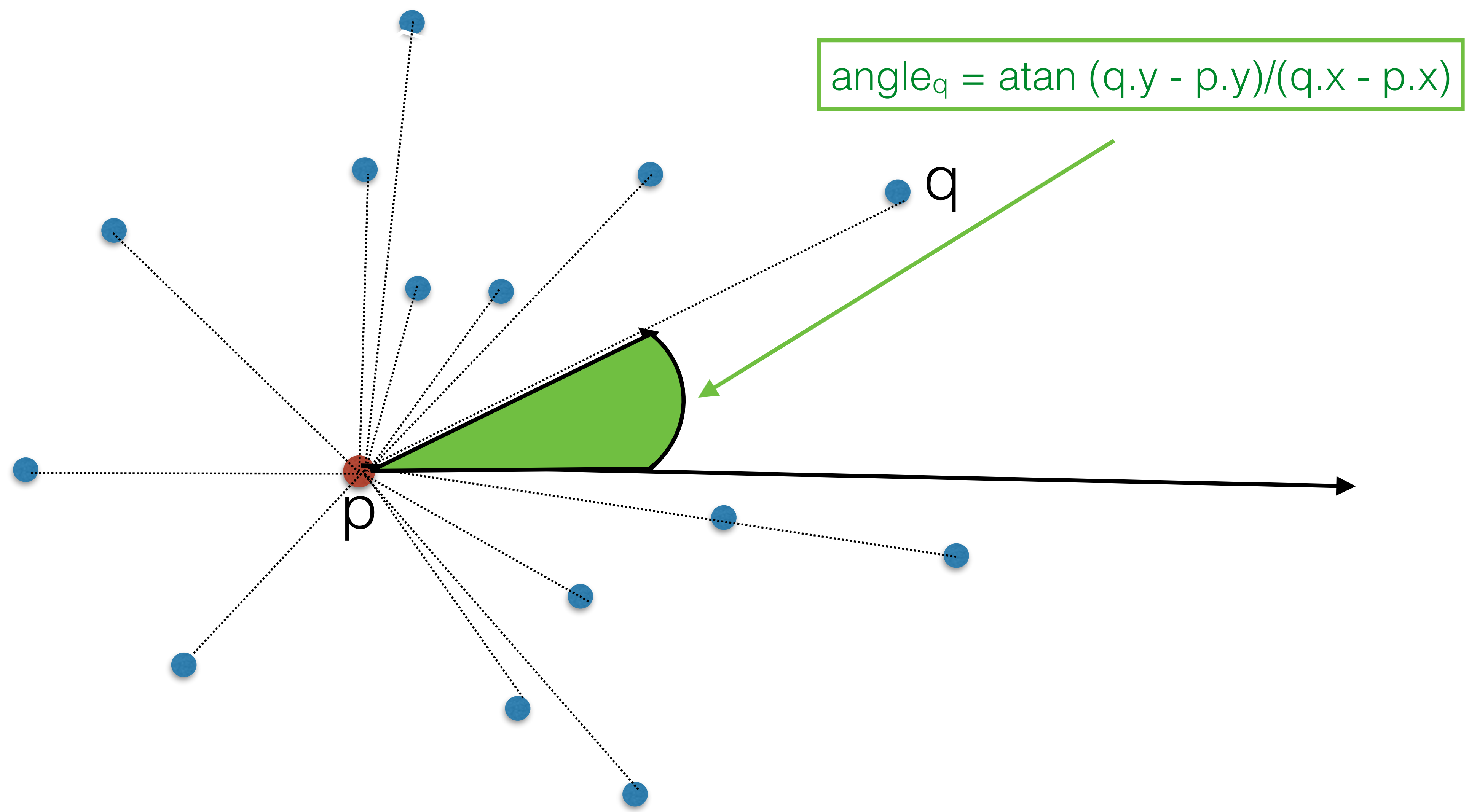
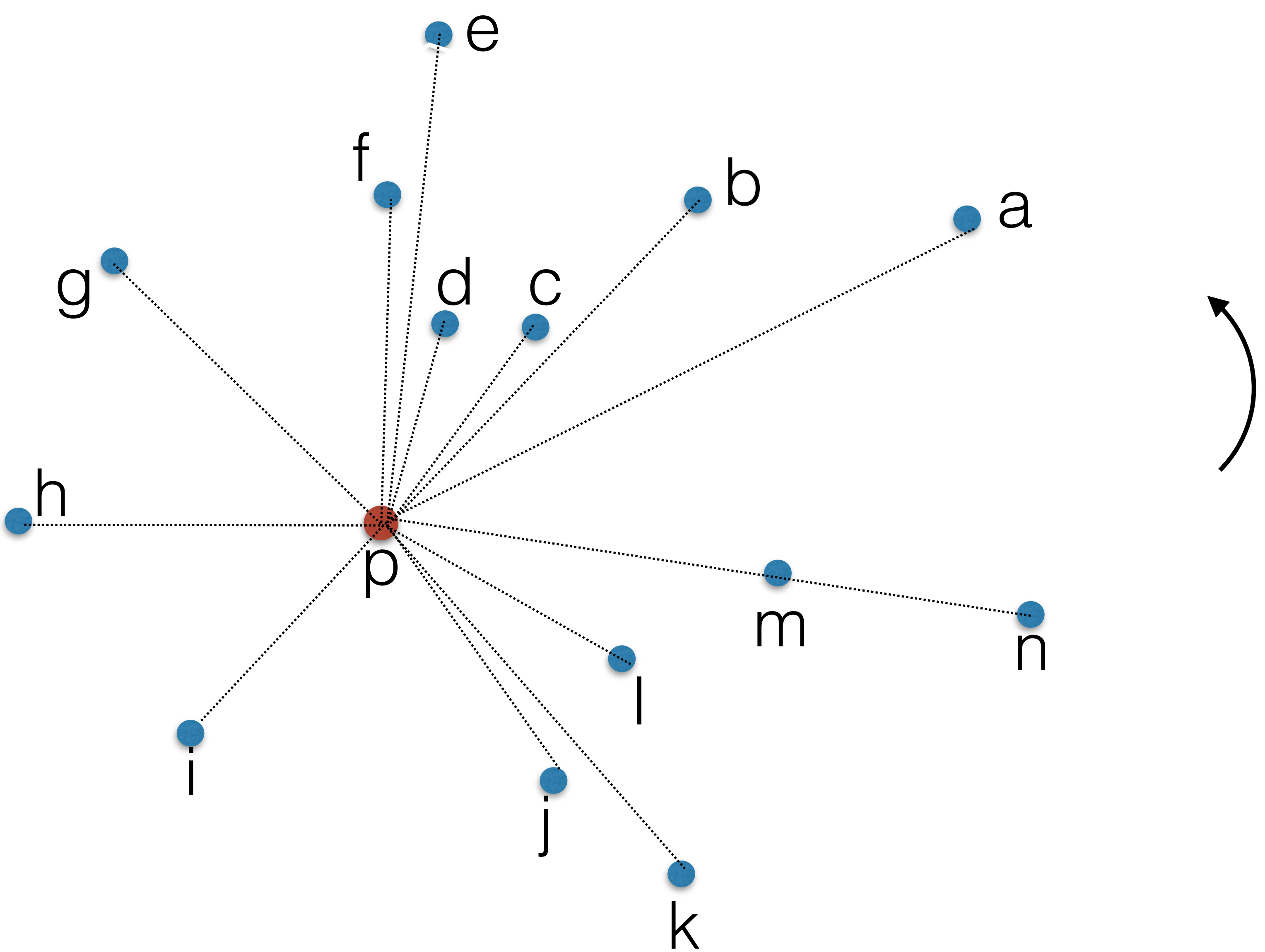


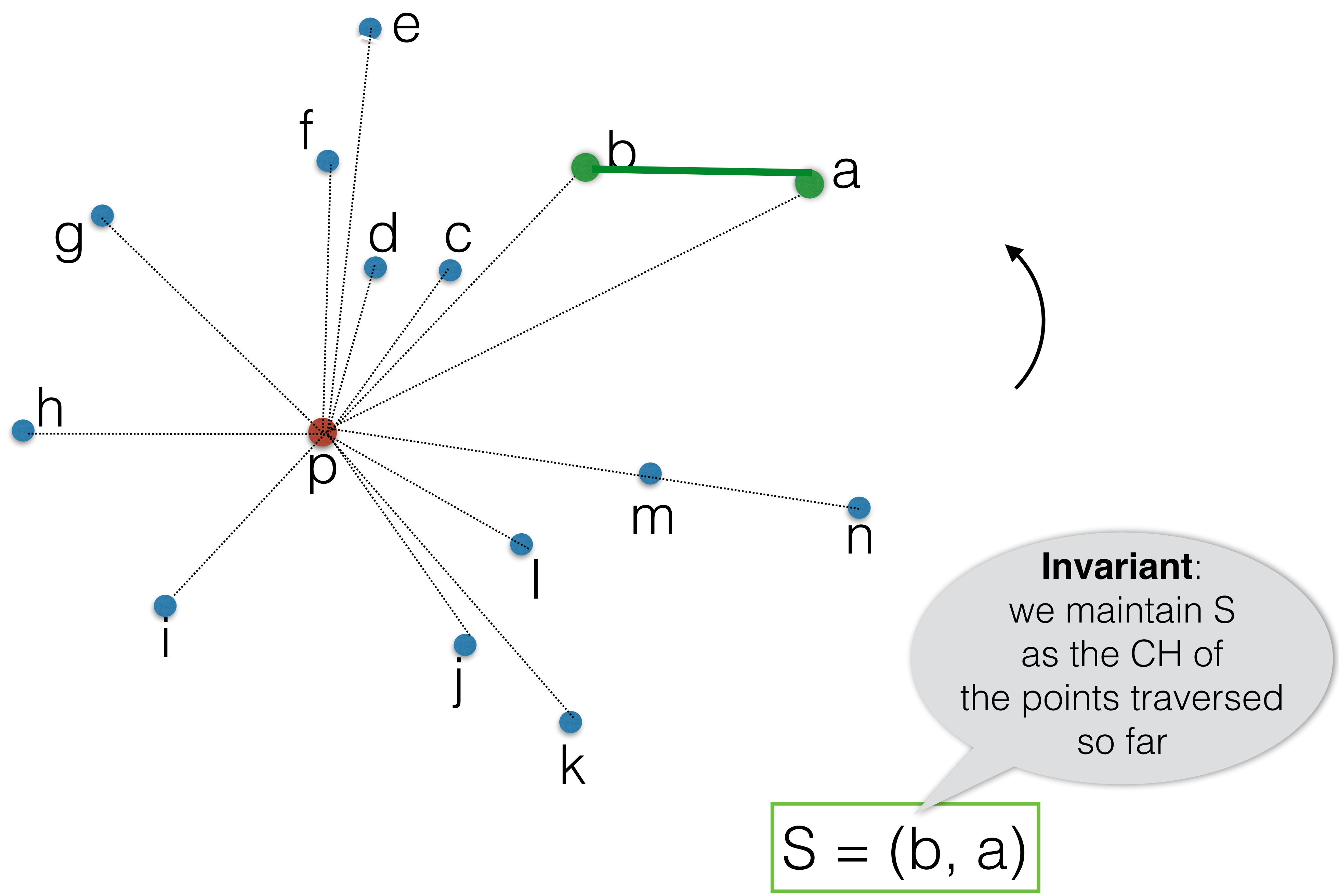
Graham scan (late 1960s)

- Idea: traverse the points in this order a, b, c, d, e, f, g,…



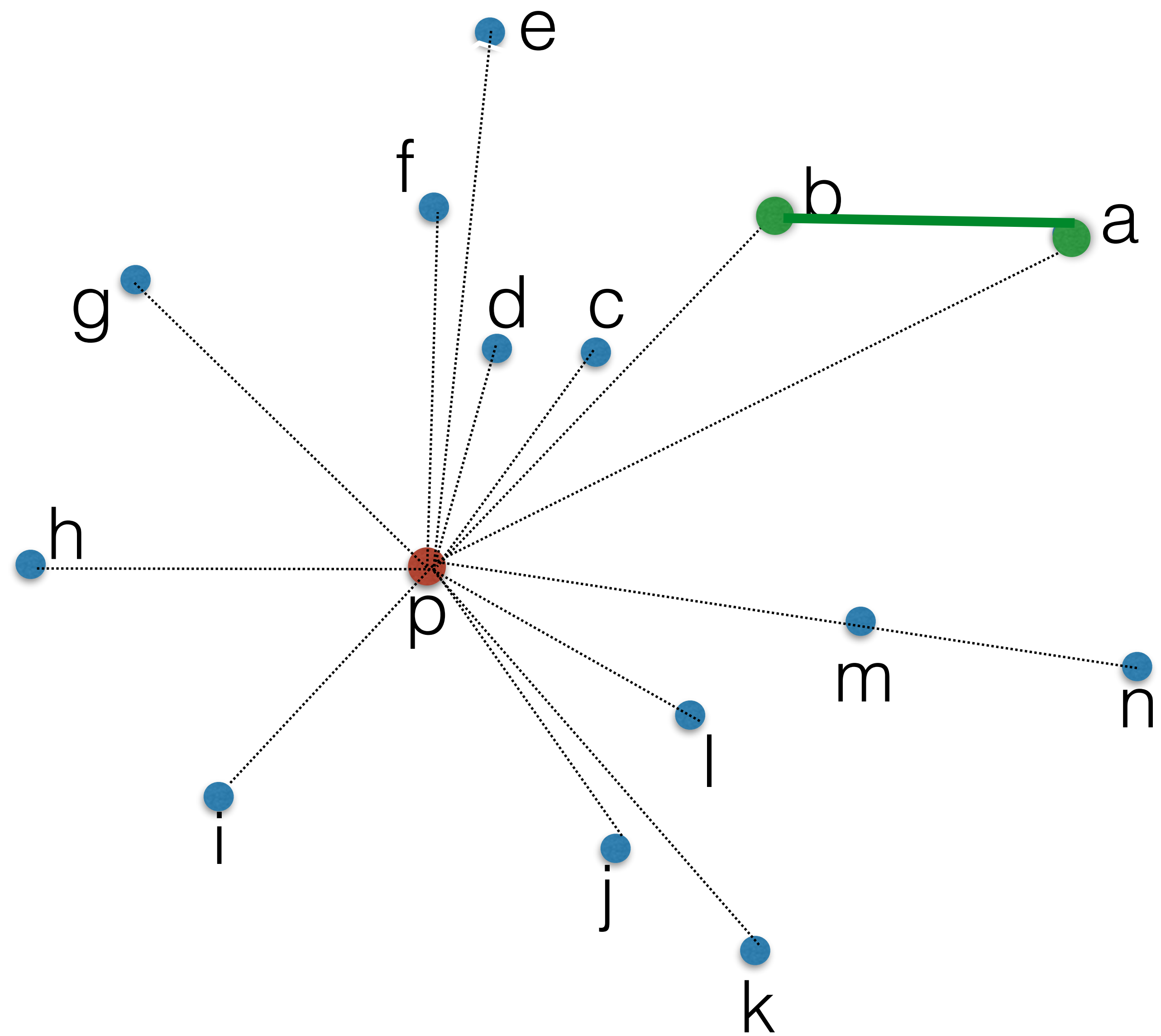
Graham scan (late 1960s)

- Idea: traverse the points in this order a, b, c, d, e, f, g,…
    - initially we put a, b in S



Graham scan (late 1960s)

Now we read point c: what do we do with it?

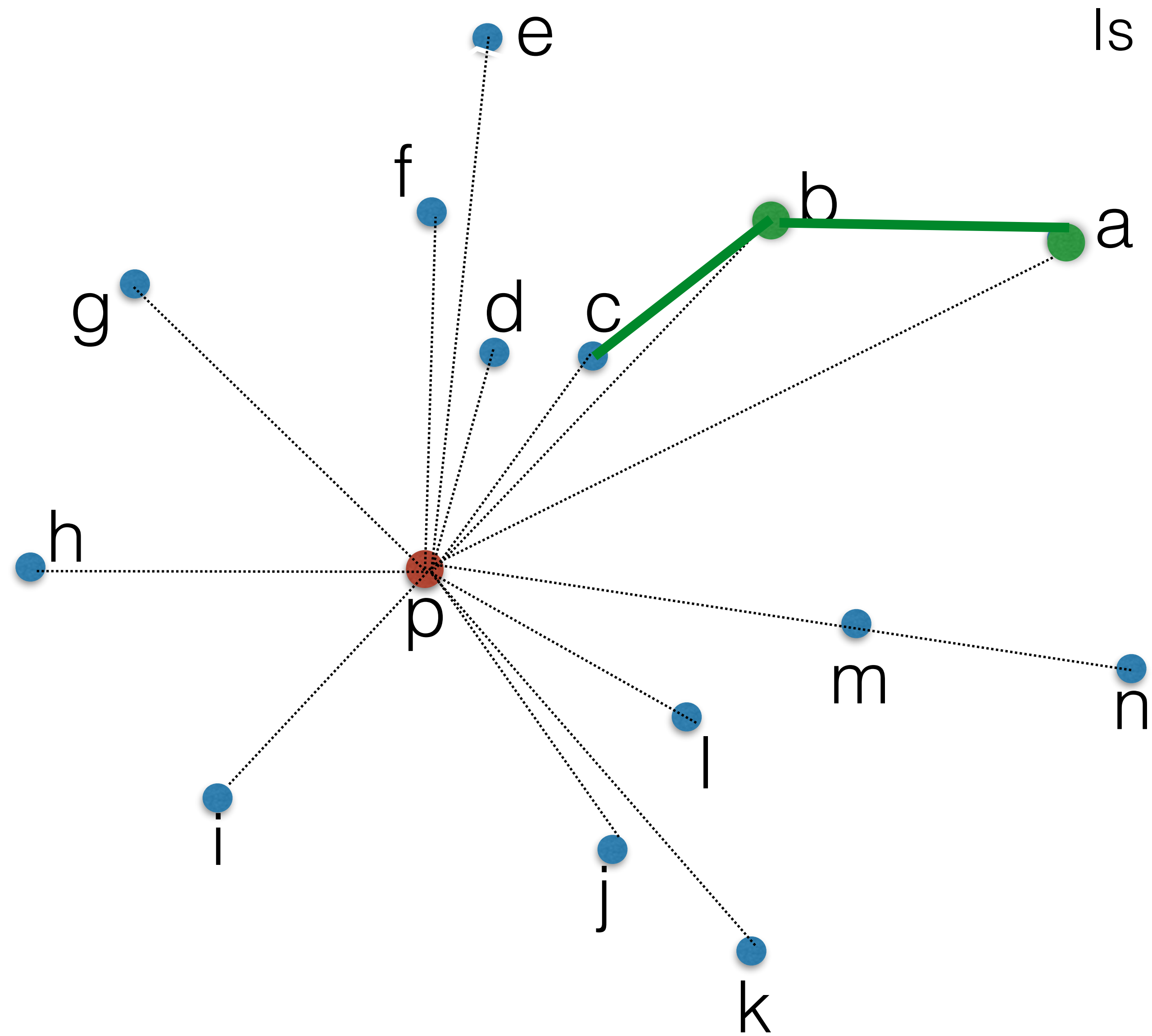


Invariant:
we maintain S
as the CH of
the points traversed
so far

$S = (b, a)$

Graham scan (late 1960s)

Now we read point c: what do we do with it?



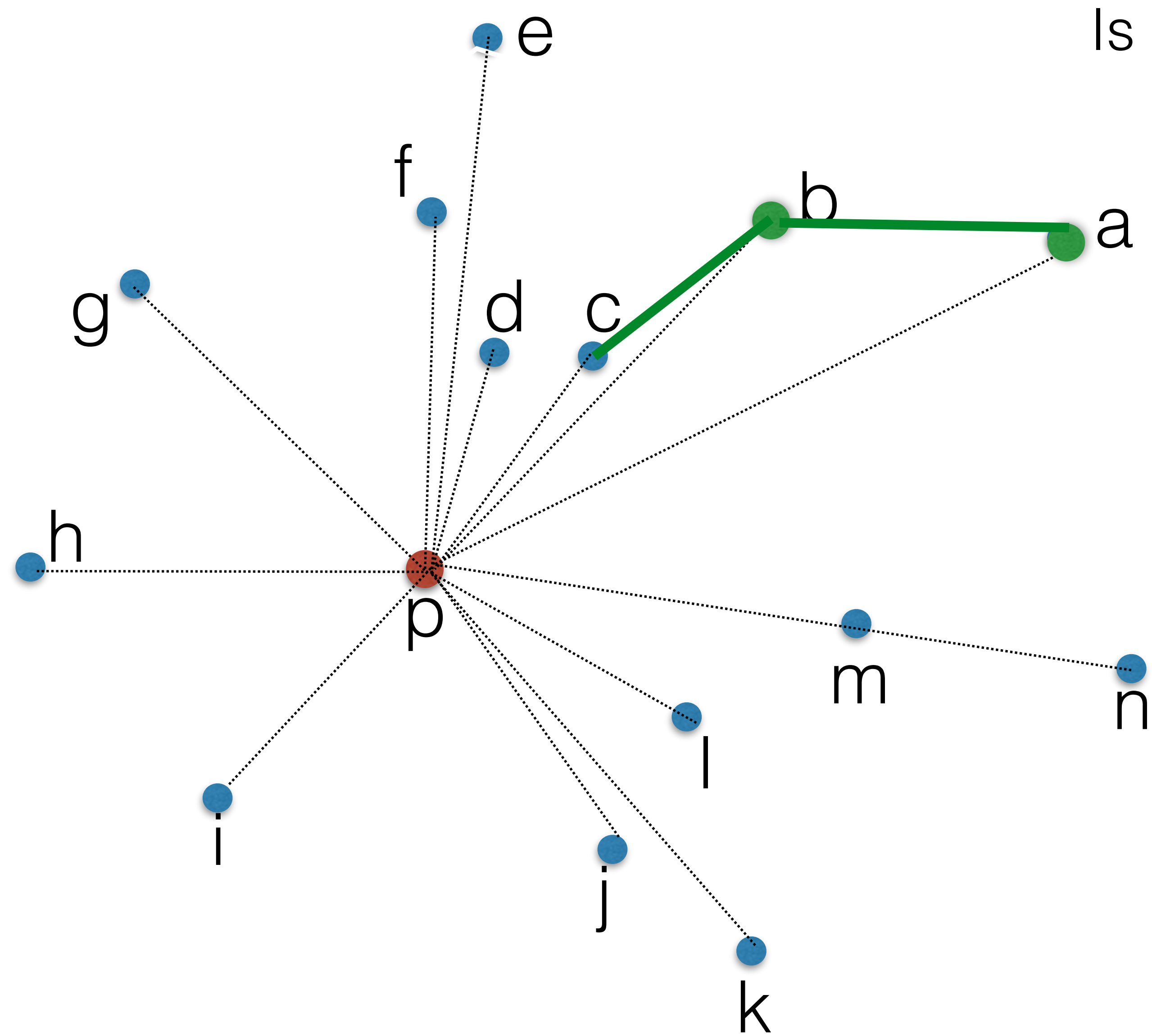
Is (c, b, a) convex?

Invariant: we maintain S as the CH of the points traversed so far

$S = (b, a)$

Graham scan (late 1960s)

Now we read point c: what do we do with it?



Is (c, b, a) convex?

YES!

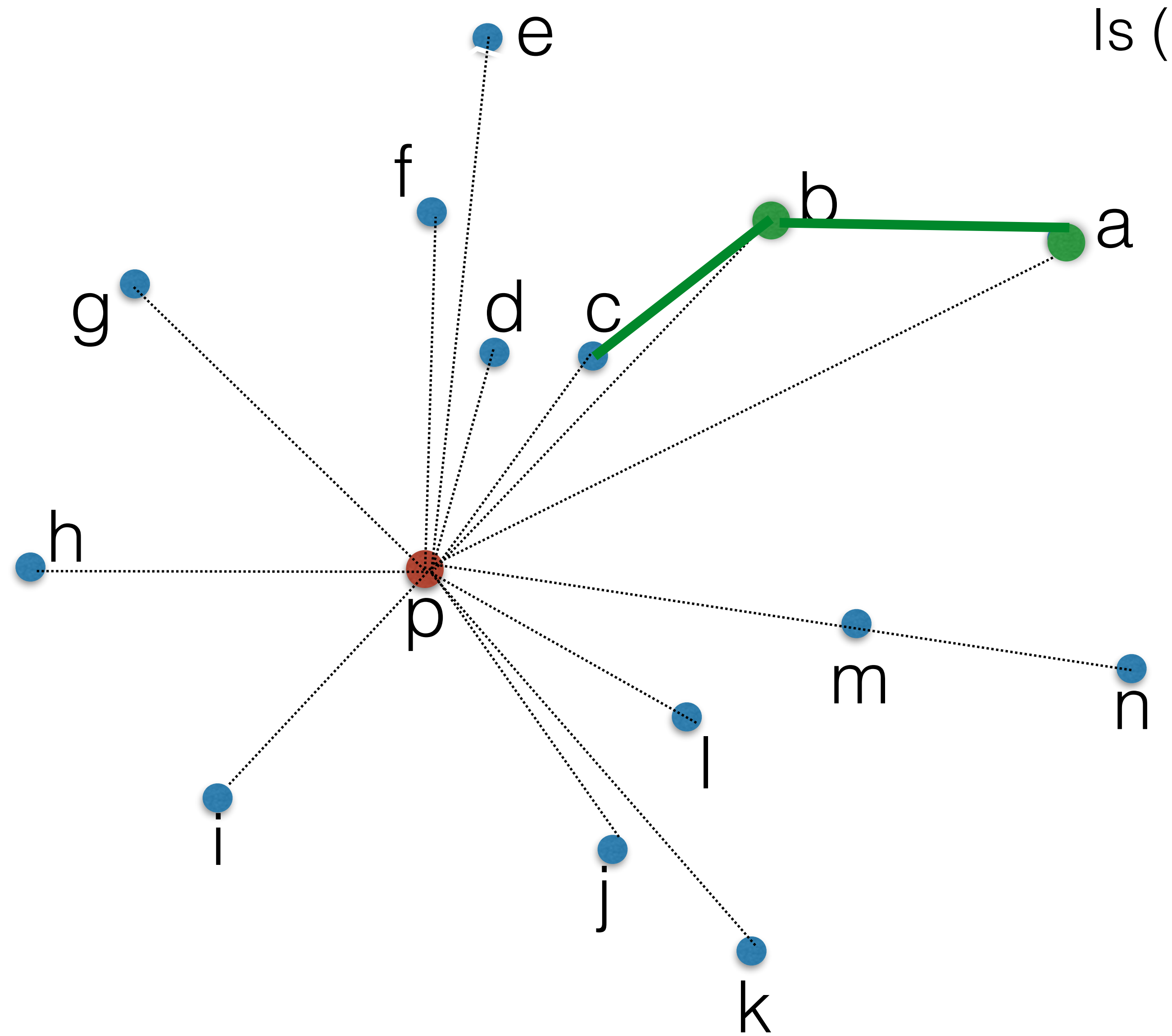
Invariant: we maintain S as the CH of the points traversed so far

S = (b, a)

Graham scan

is c left of ab

Now we read point c: if (c+S) stays convex : add c to S



Is (c, b, a) convex?

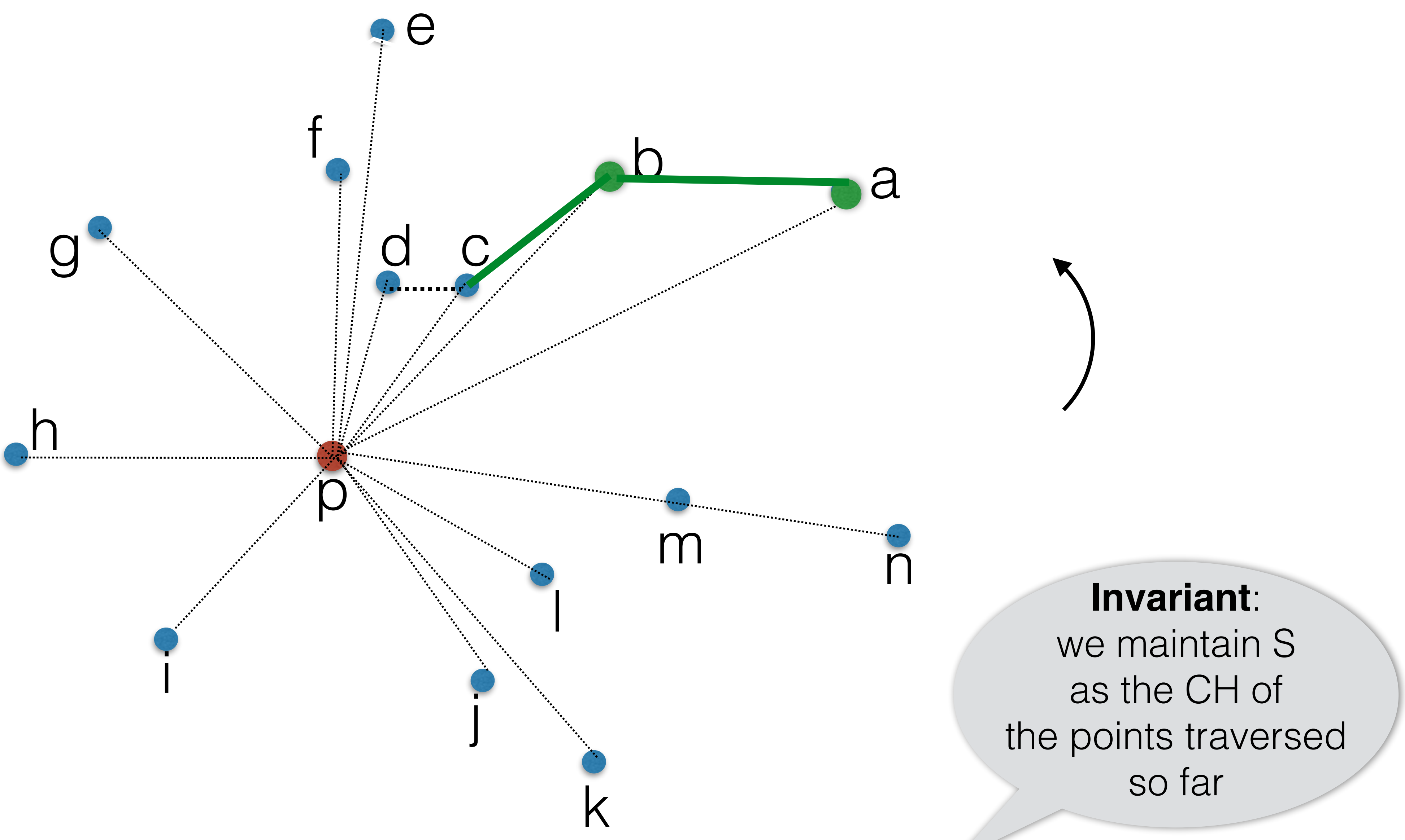
YES!

Invariant: we maintain S as the CH of the points traversed so far

S = (c, b, a)

Graham scan (late 1960s)

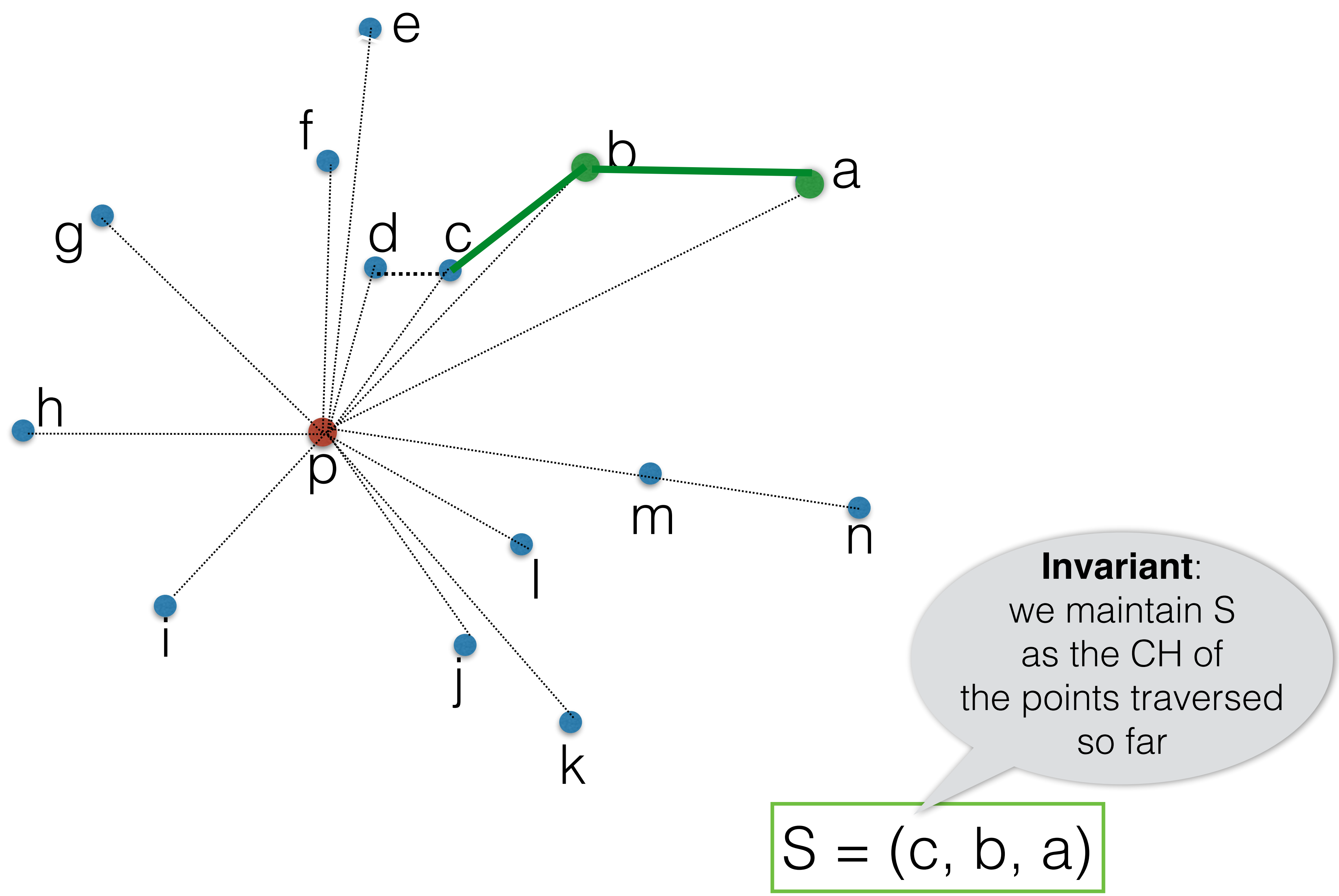
Now we read point d:



$S = (c, b, a)$

Graham scan (late 1960s)

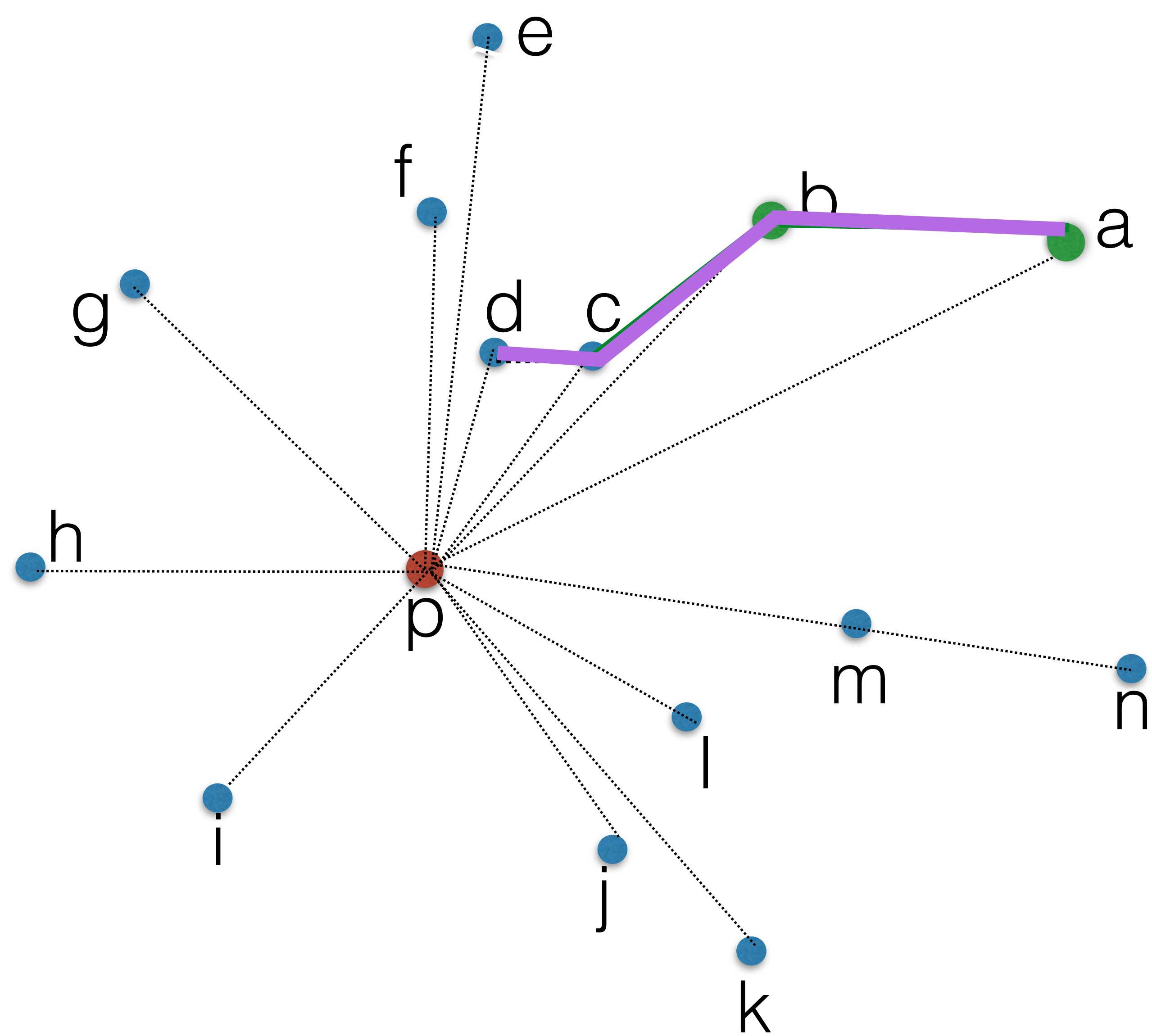
Now we read point d: is d left of bc? NO



Graham scan (late 1960s)

Now we read point d: is d left of bc? NO

//can't add d, because (d,c,b,a) not convex



Invariant:
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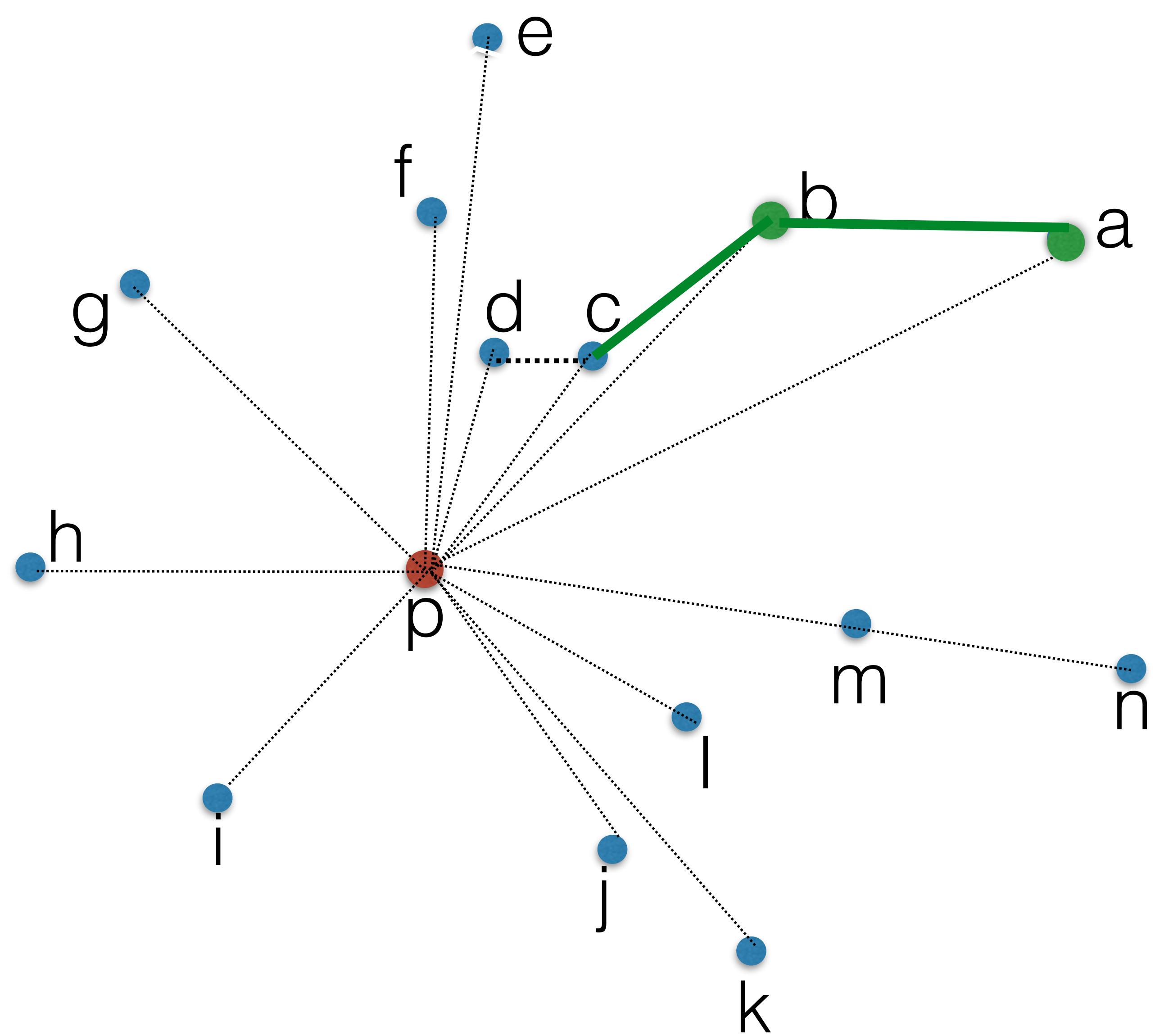
S = (c, b, a)

Graham scan (late 1960s)

Now we read point d: is d left of bc? NO



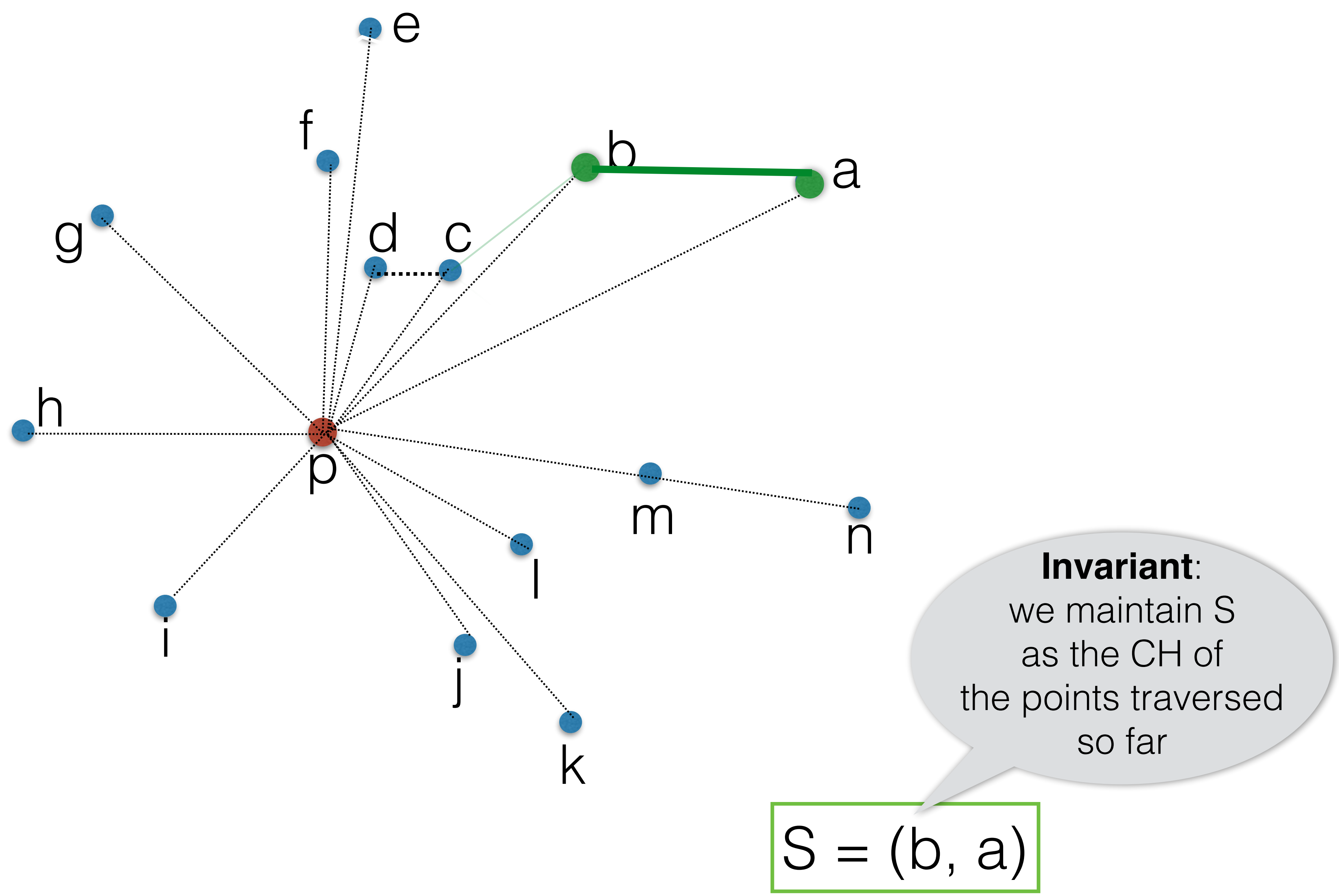
Invariant:
we maintain S
as the CH of
the points traversed
so far

S = (c, b, a)

Graham scan (late 1960s)

Now we read point d: is d left of bc? NO

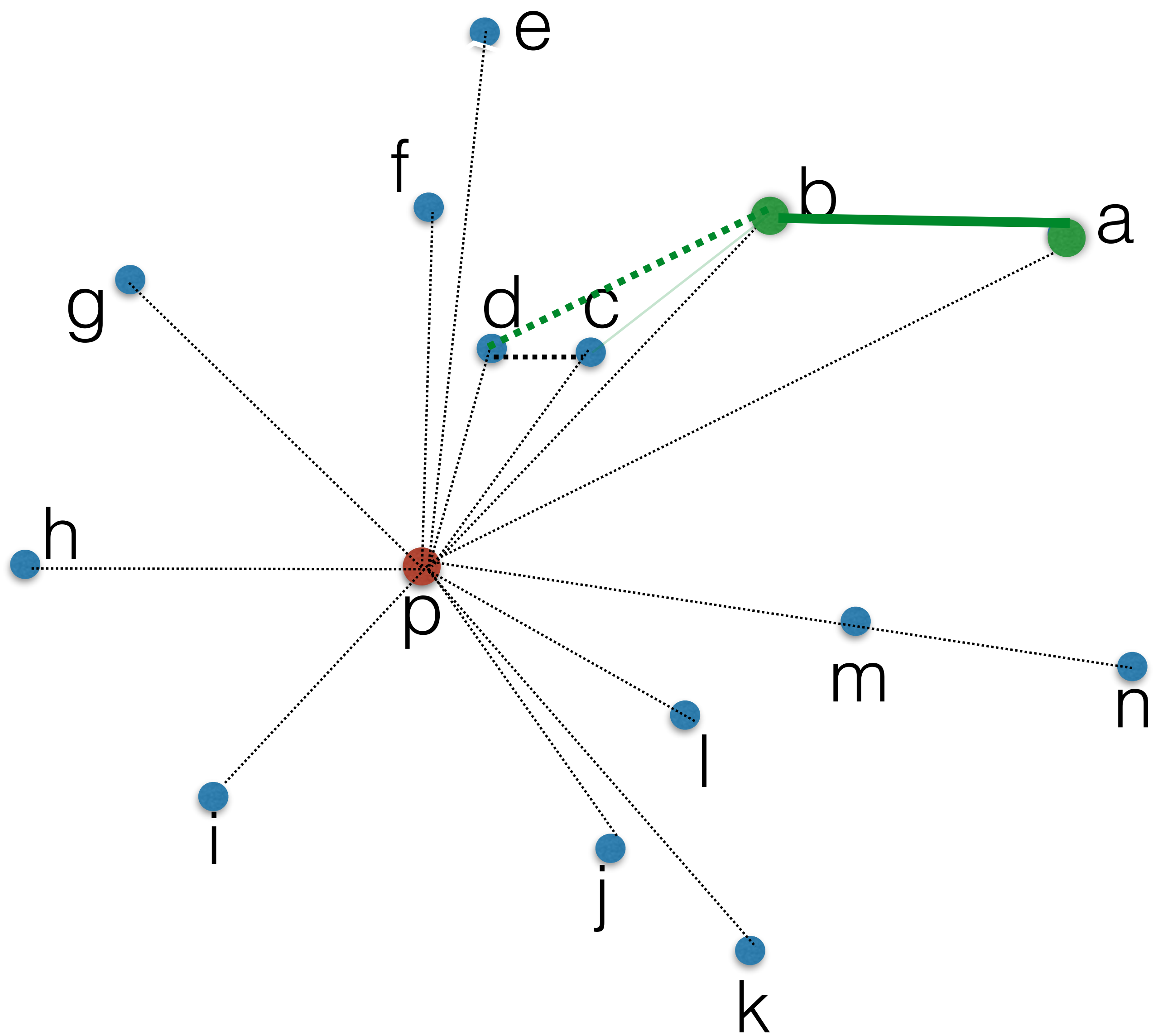
pop c; is d left of ab?



Graham scan (late 1960s)

Now we read point d: is d left of bc? NO

pop c; is d left of ab?



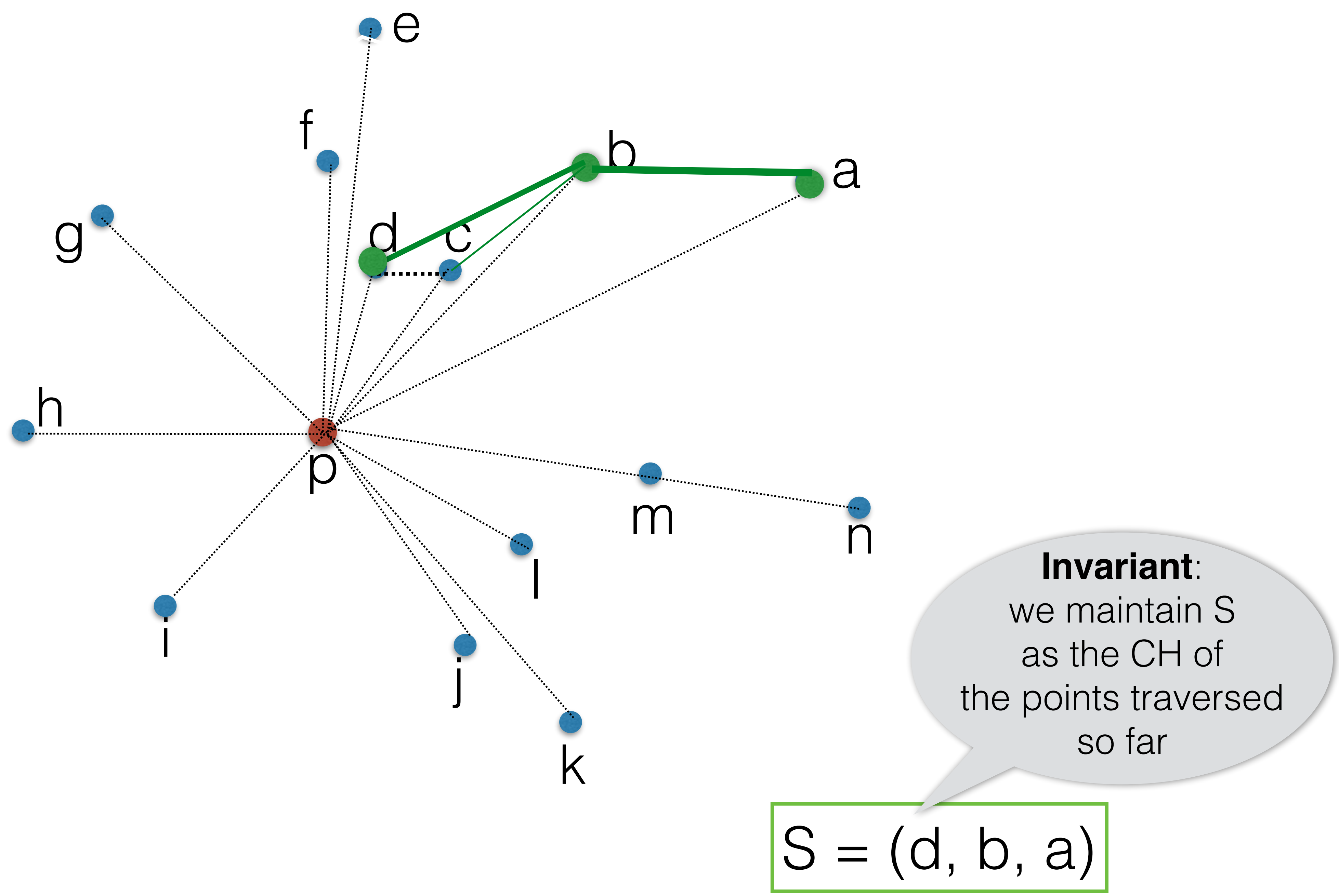
Invariant:
we maintain S
as the CH of
the points traversed
so far

S = (b, a)

Graham scan (late 1960s)

Now we read point d: is d left of bc? NO

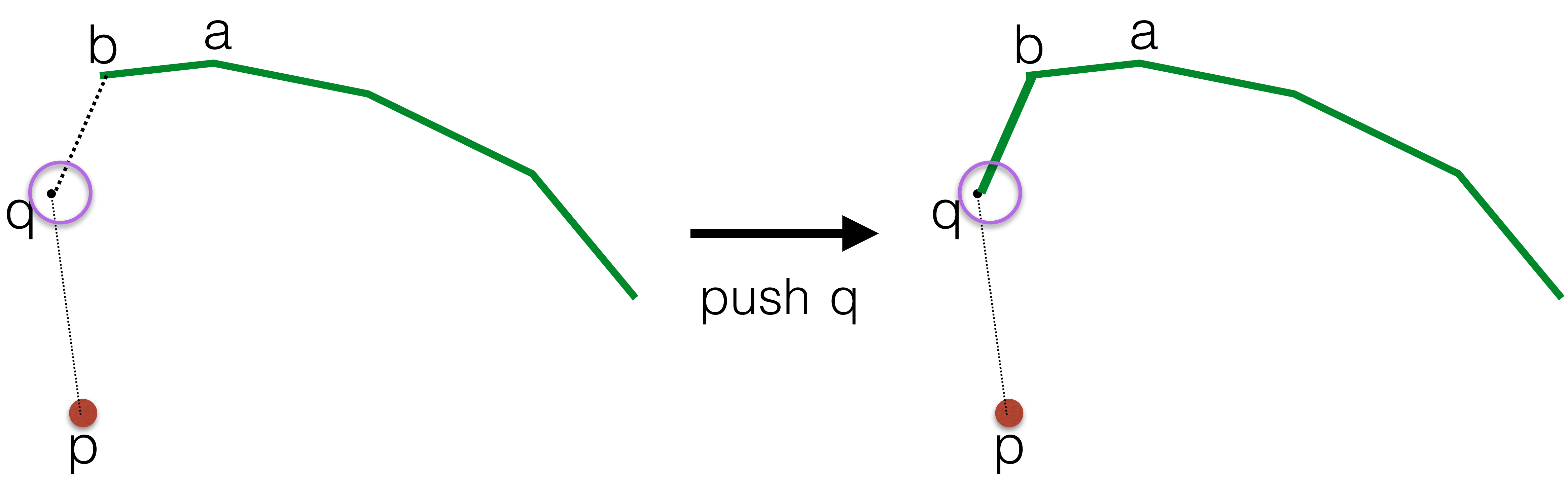
pop c; is d left of ab? YES ==> insert d in S



Graham scan (late 1960s)

In general, we read next point q:

- let b = head(S), a = next(b)
- if q is left of ab: add q to S



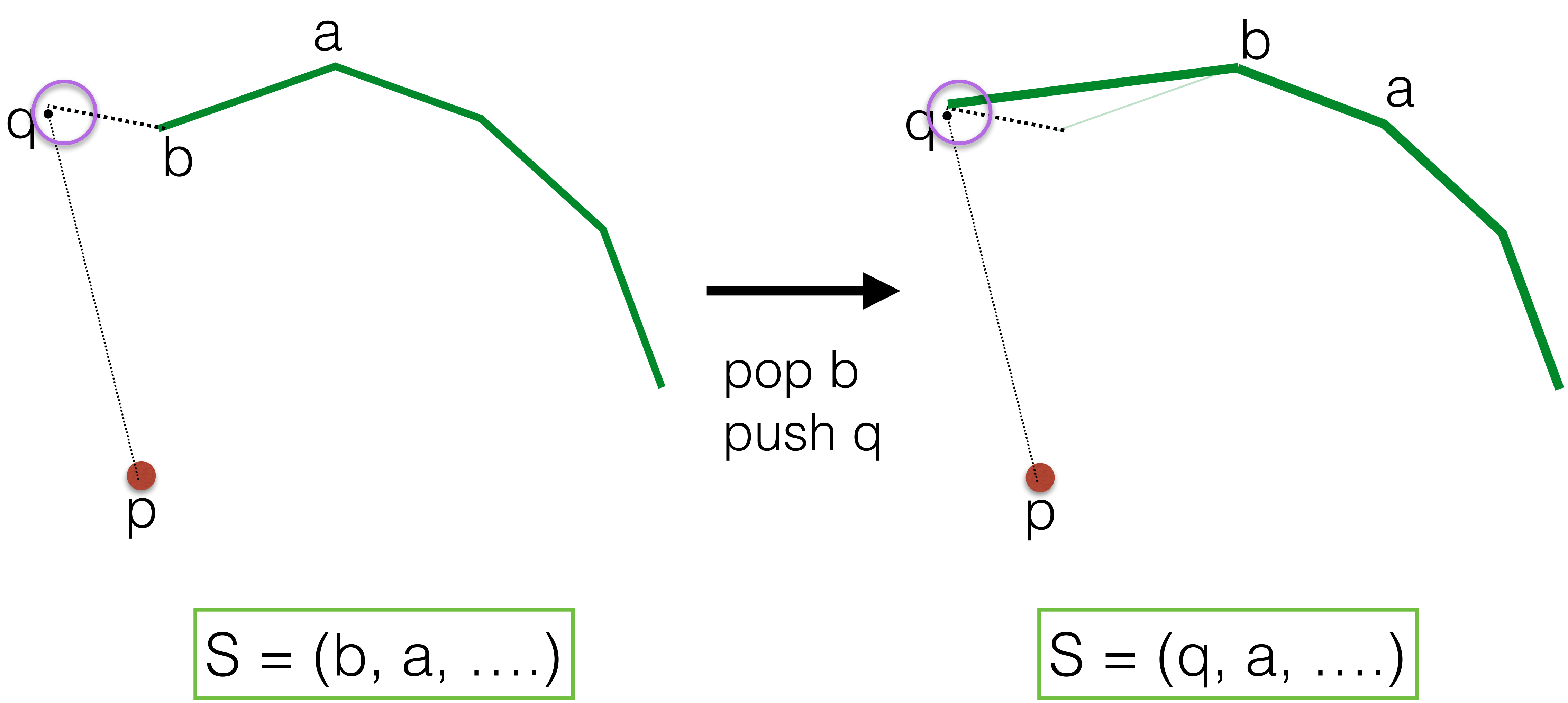
S = (b, a, ….)

S = (q, b, a, ….)

Graham scan (late 1960s)

In general, we read next point q:

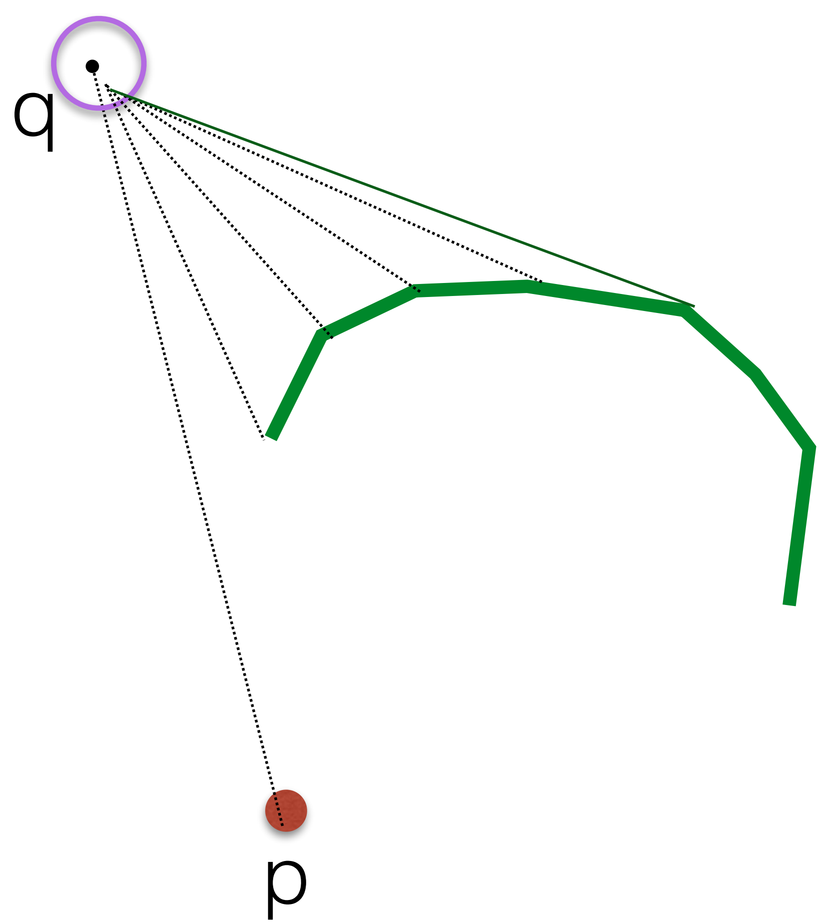
- let b = head(S), a = next(b)
- if q is right of ab: pop b; repeat until q is left of ab, then add q to S

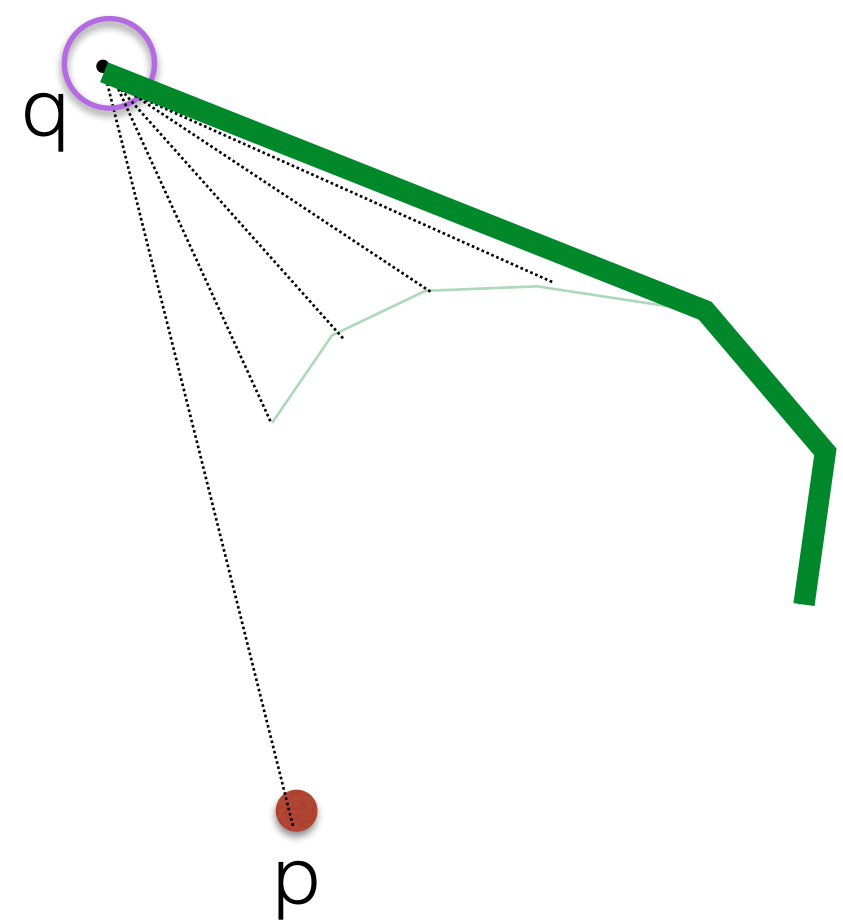


How many vertices might need to be popped, when looking at the next vertex q?

Graham scan (late 1960s)

Cascading pops





Graham scan: ANALYSIS

- call them p_1, p_2, p_3, …p_{n-1} in this order

- Find interior point p_0
- Sort all other points ccw around p_0…..
- Initialize stack S = (p_2, p_1)
- for i=3 to n-1 do
  - if p_i is left of (second(S),first(S)):
    - push p_i on S
  - else
    - do
      - pop S
    - while p_i is right of (second(S), first(S))
    - push p_i on S

note that we are ignoring some details, such as what happens if first point p1 is not on the CH, and can the stack ever run empty. We'll see we can avoid both.

Graham scan: ANALYSIS

- Find interior point p_0
- Sort all other points ccw around p_0…..
- Initialize stack S = (p_2, p_1)
- for i=3 to n-1 do
  - if p_i is left of (second(S),first(S)):
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    - do
      - pop S
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    - push p_i on S

O(n) (we'll think of it later)

Graham scan: ANALYSIS

- Find interior point p_0
- Sort all other points ccw around p_0…..
- Initialize stack S = (p_2, p_1)
- for i=3 to n-1 do
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      - pop S
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O(n) (we'll think of it later)

O(n lg n)

Graham scan: ANALYSIS

- Find interior point p_0
- Sort all other points ccw around p_0.....
- Initialize stack S = (p_2, p_1)
- for i=3 to n-1 do
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  - else
    - do
      - pop S
    - while p_i is right of (second(S), first(S))
    - push p_i on S

O(n) (we'll think of it later)

O(n lg n)

How long does this take?

Graham scan: ANALYSIS

- Find interior point p_0
- Sort all other points ccw around p_0.....
- Initialize stack S = (p_2, p_1)
- for i=3 to n-1 do
  - if p_i is left of (second(S),first(S)):
    - push p_i on S
  - else
    - do
      - pop S
    - while p_i is right of (second(S), first(S))
    - push p_i on S

O(n) (we'll think of it later)

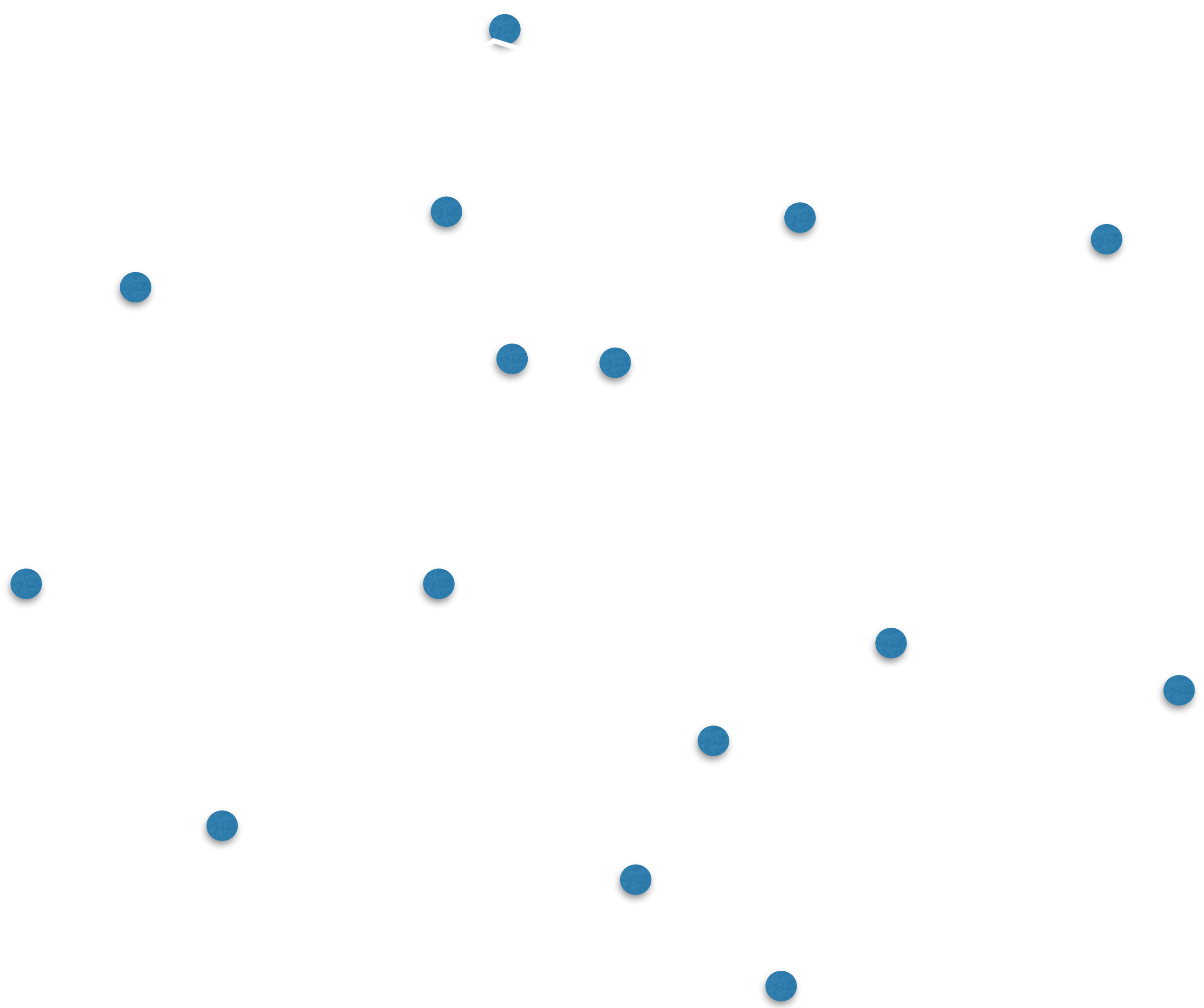
O(n lg n)

How long does this take?

every point is pushed once
and popped at most once => O(n)

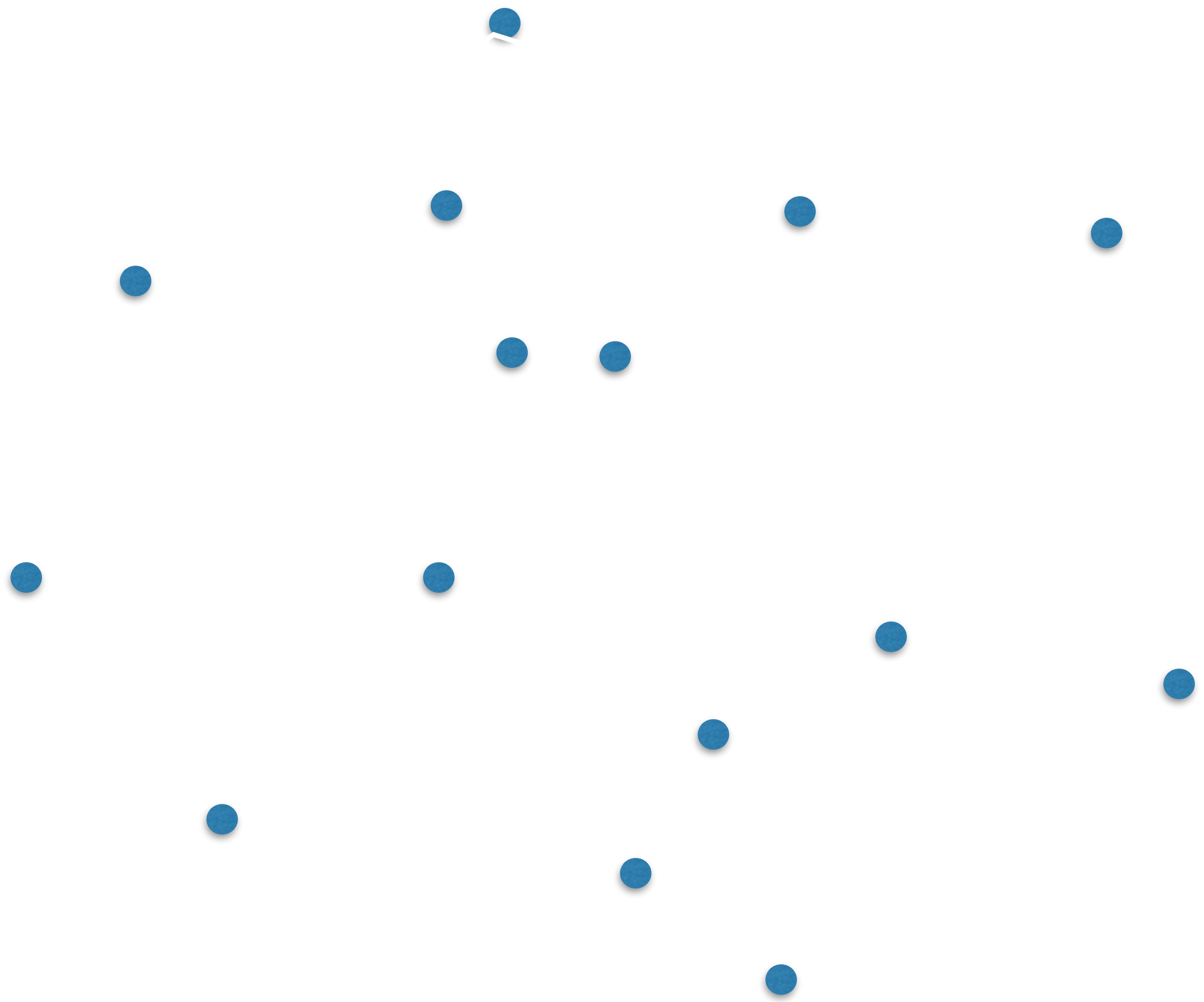
Graham scan: Details

- How to find an interior point?



Graham scan: Details

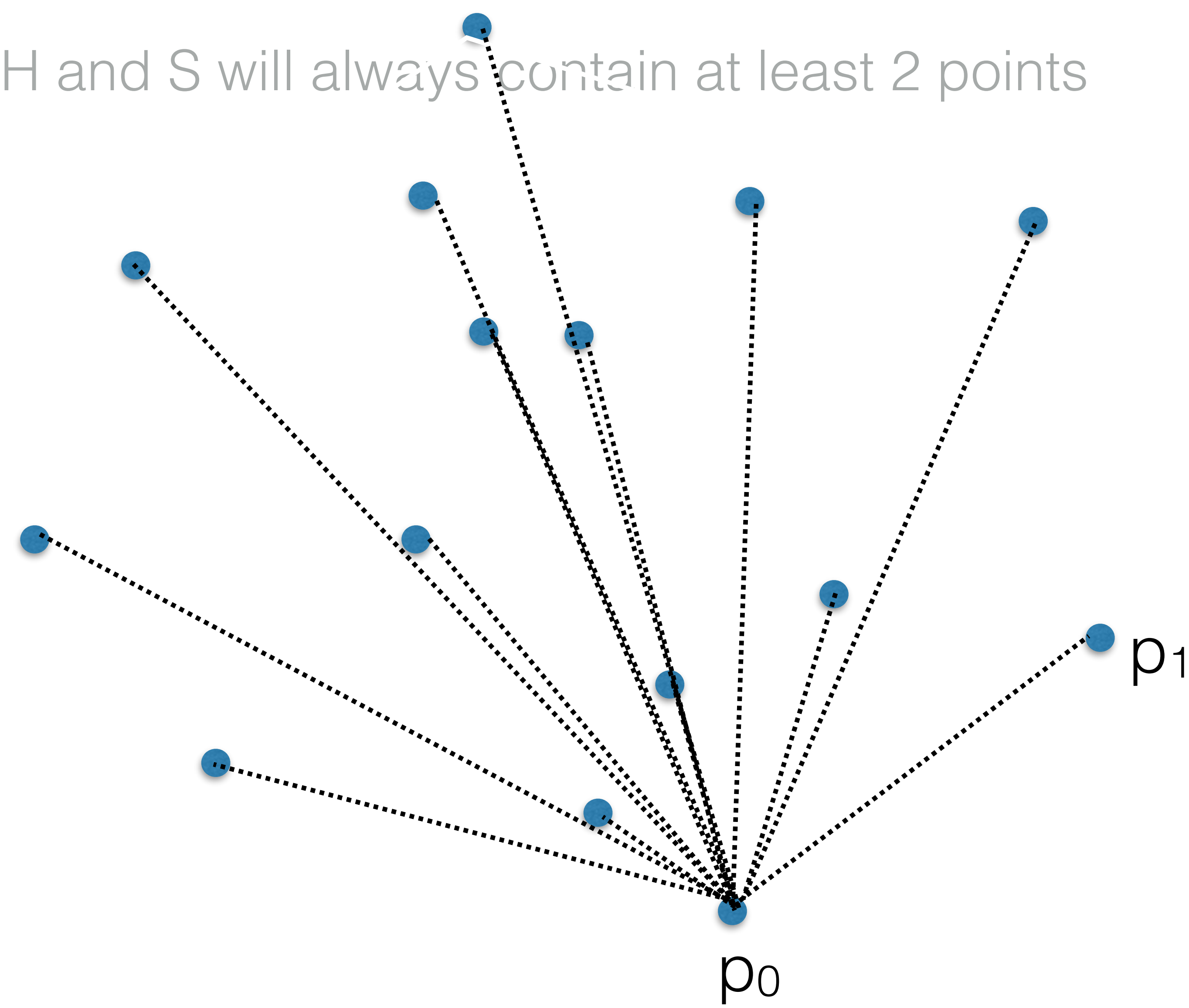
- How to find an interior point?
- A simplification is to pick p_0 as the lowest point



Graham scan: Details

- How to find an interior point?
- A simplification is to pick p_0 as the lowest point
  - initialize stack S = (p1, p0)

//both are on CH and S will always contain at least 2 points



Graham scan: Class work

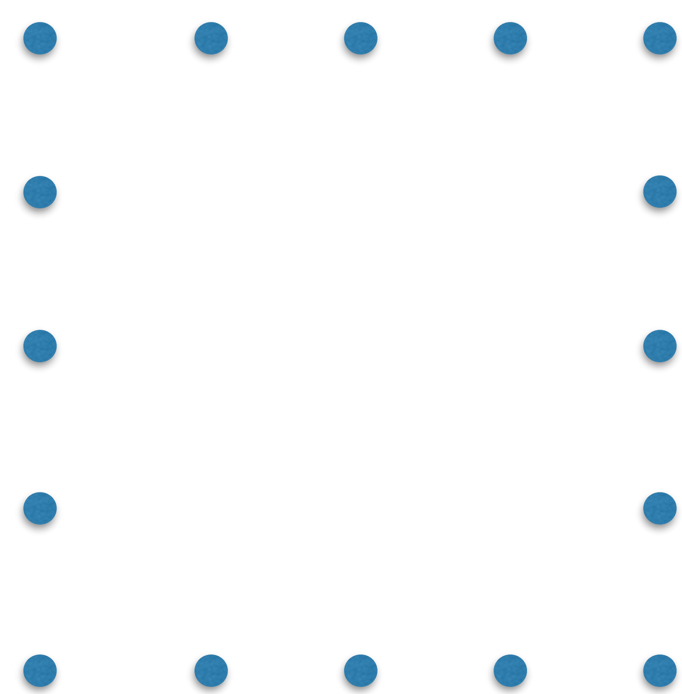
- Choose a set of “interesting” points and go through the algorithm

- Think what constitute degenerate cases. Does the algorithm handle them? If not, how do you fix it?

Graham scan: Details

- Handling collinear-ities

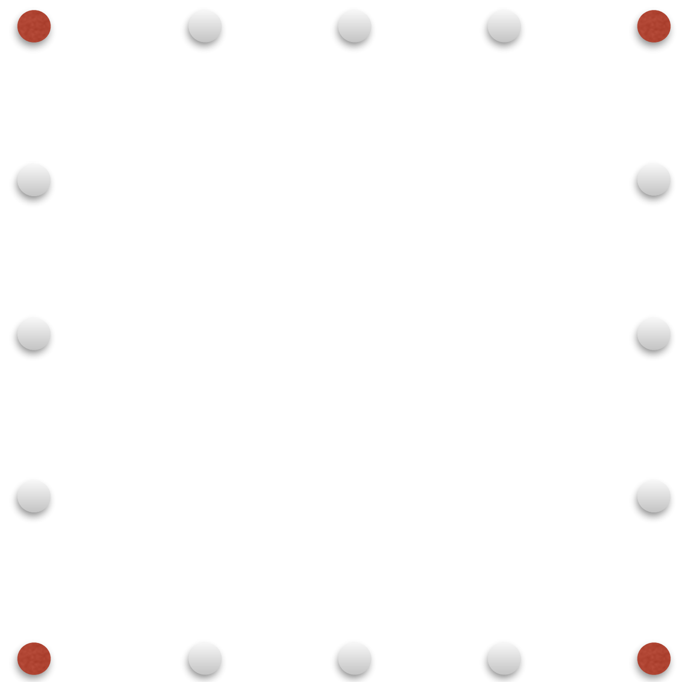
What happens when you run on this input?



How can you fix it?

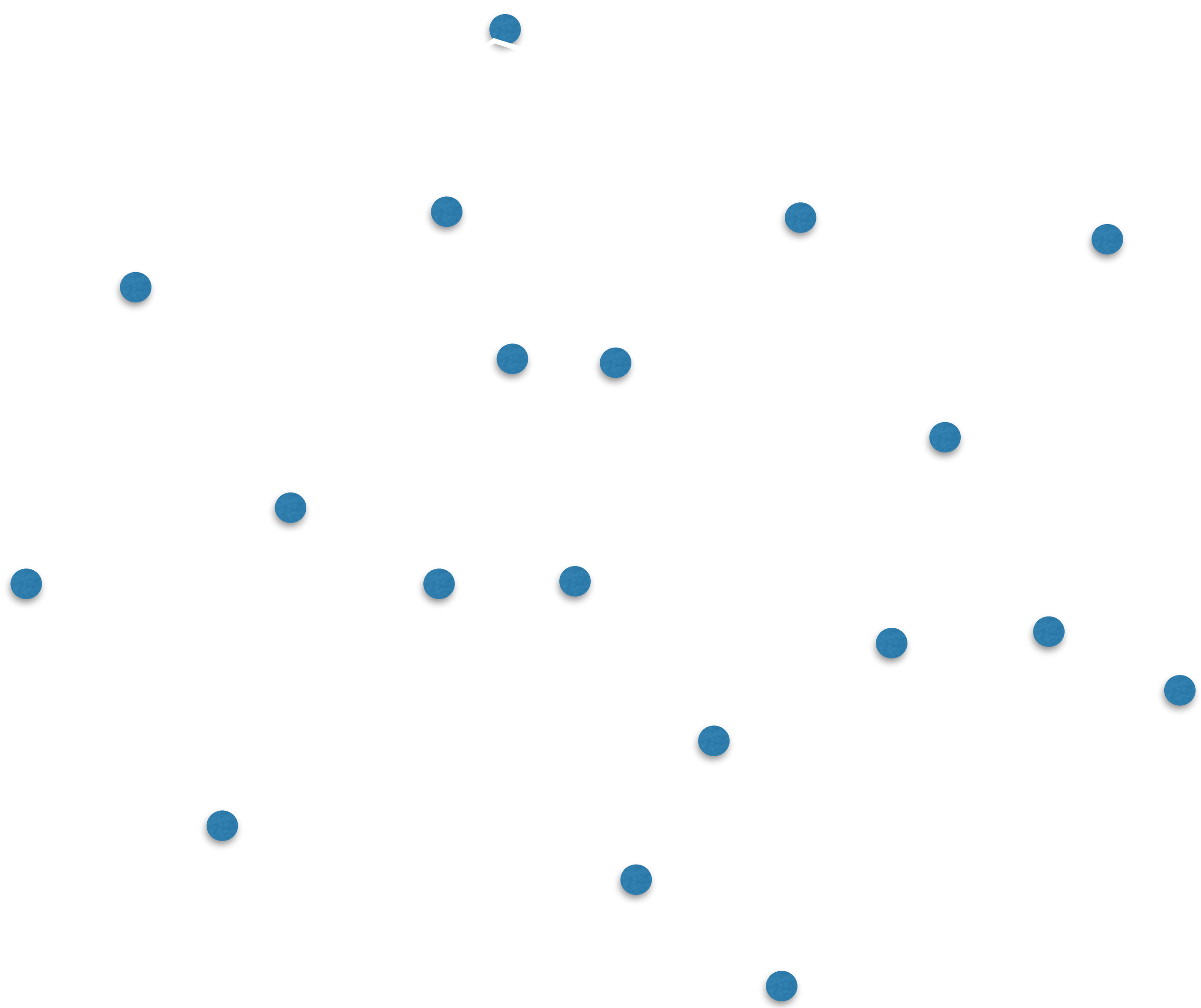
Graham scan: Details

- Handling collinear-ities

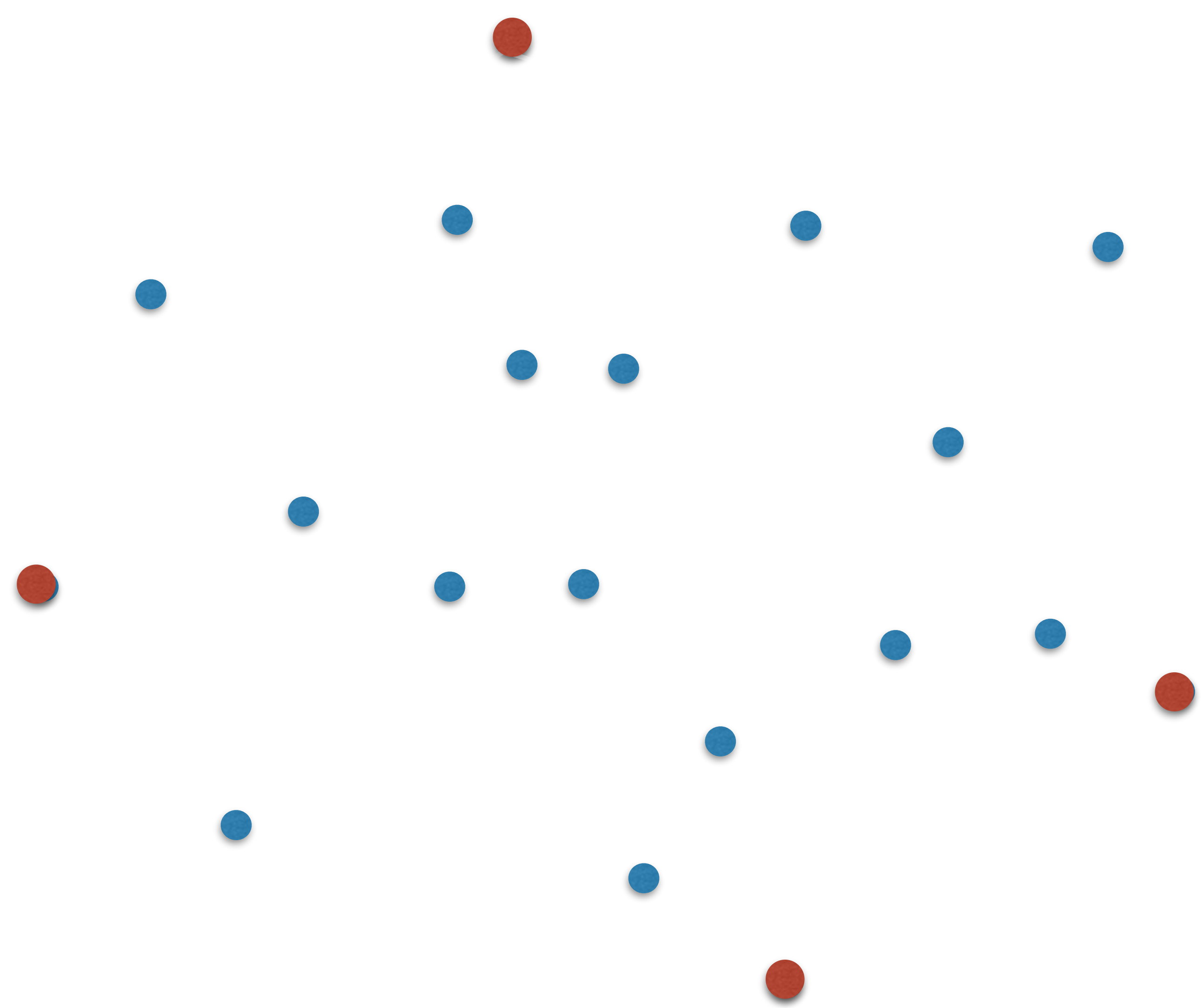


This is what we want:

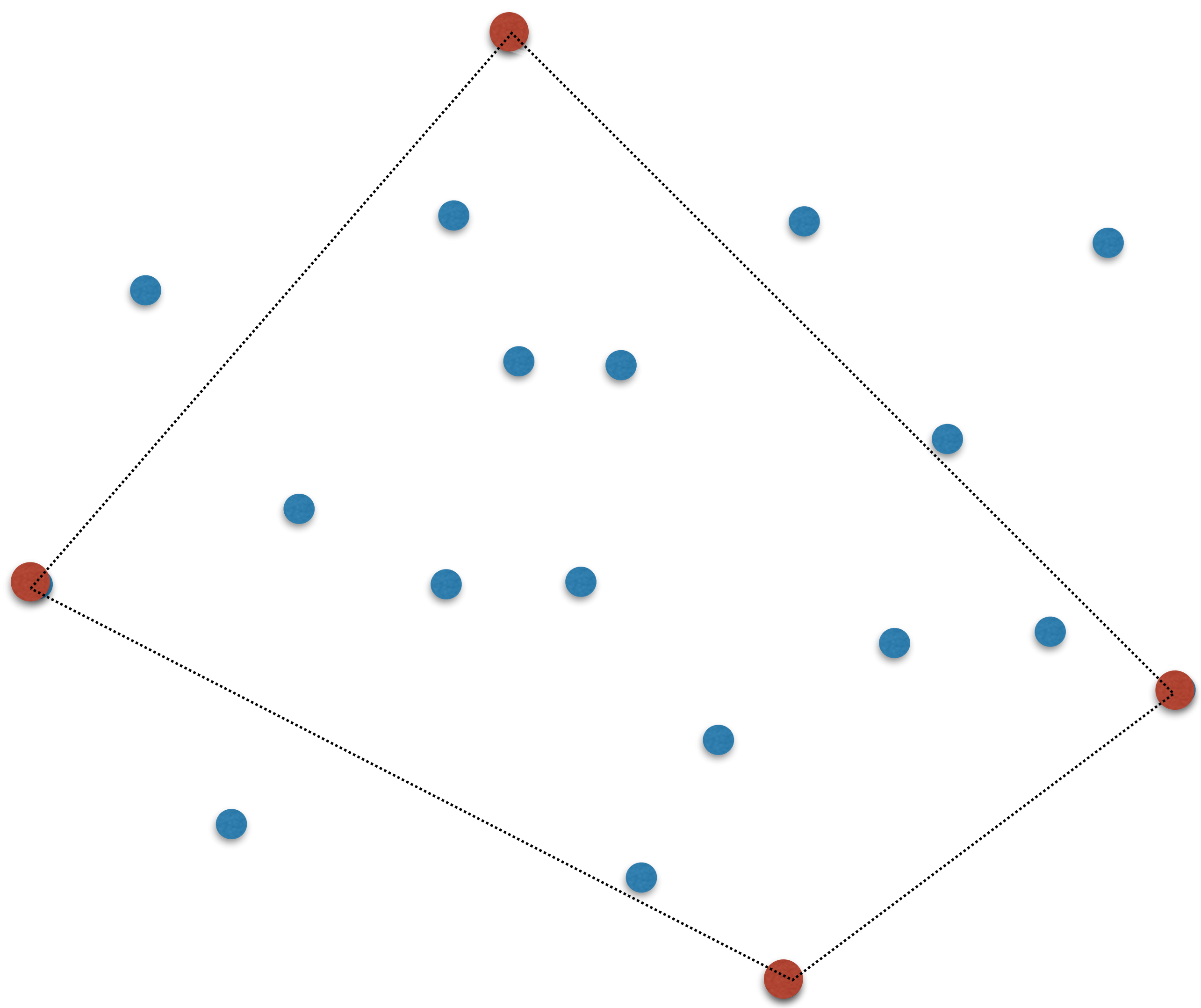
Speeding up Graham scan



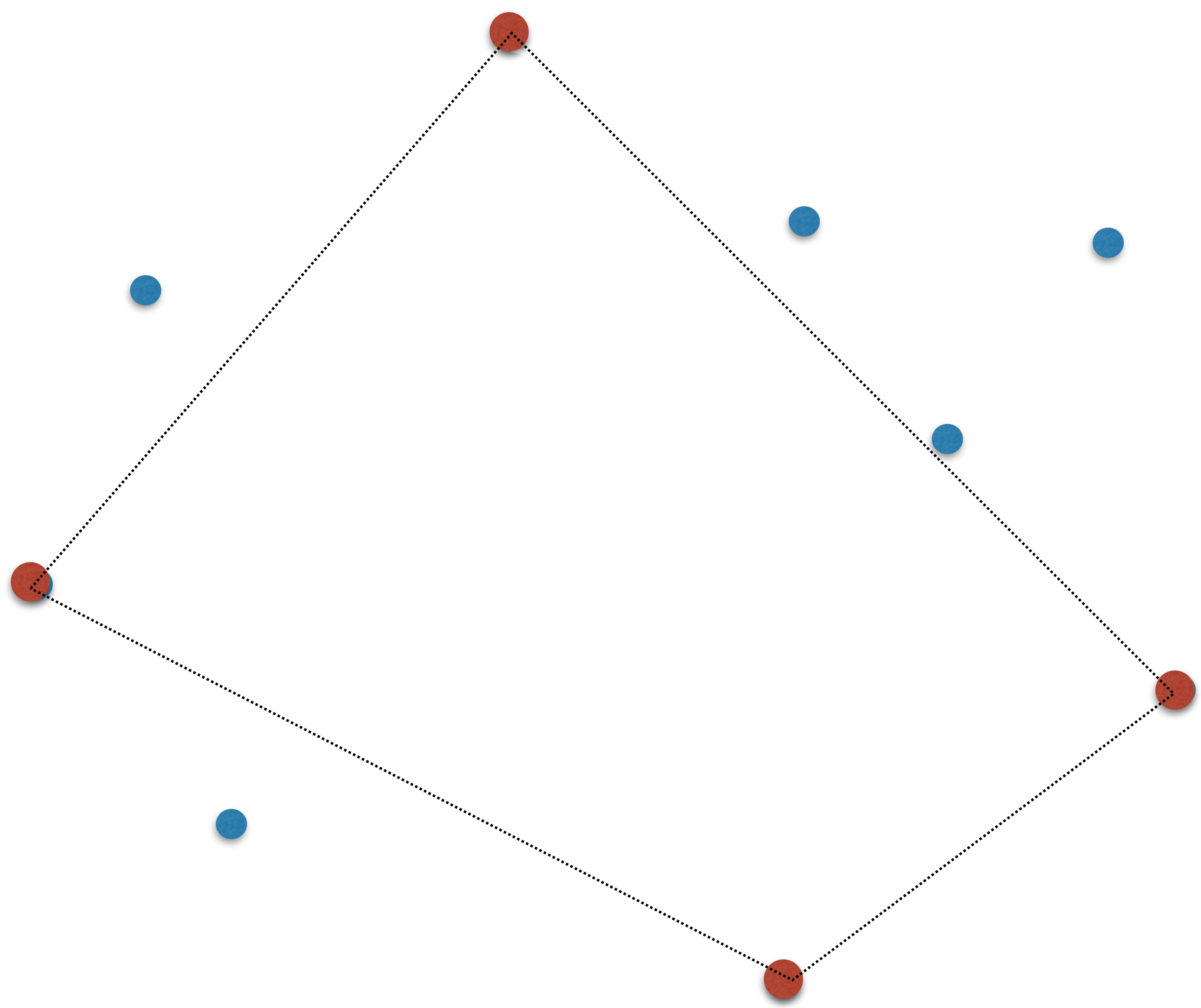
Speeding up Graham scan



Speeding up Graham scan



Speeding up Graham scan



Speeding up Graham scan