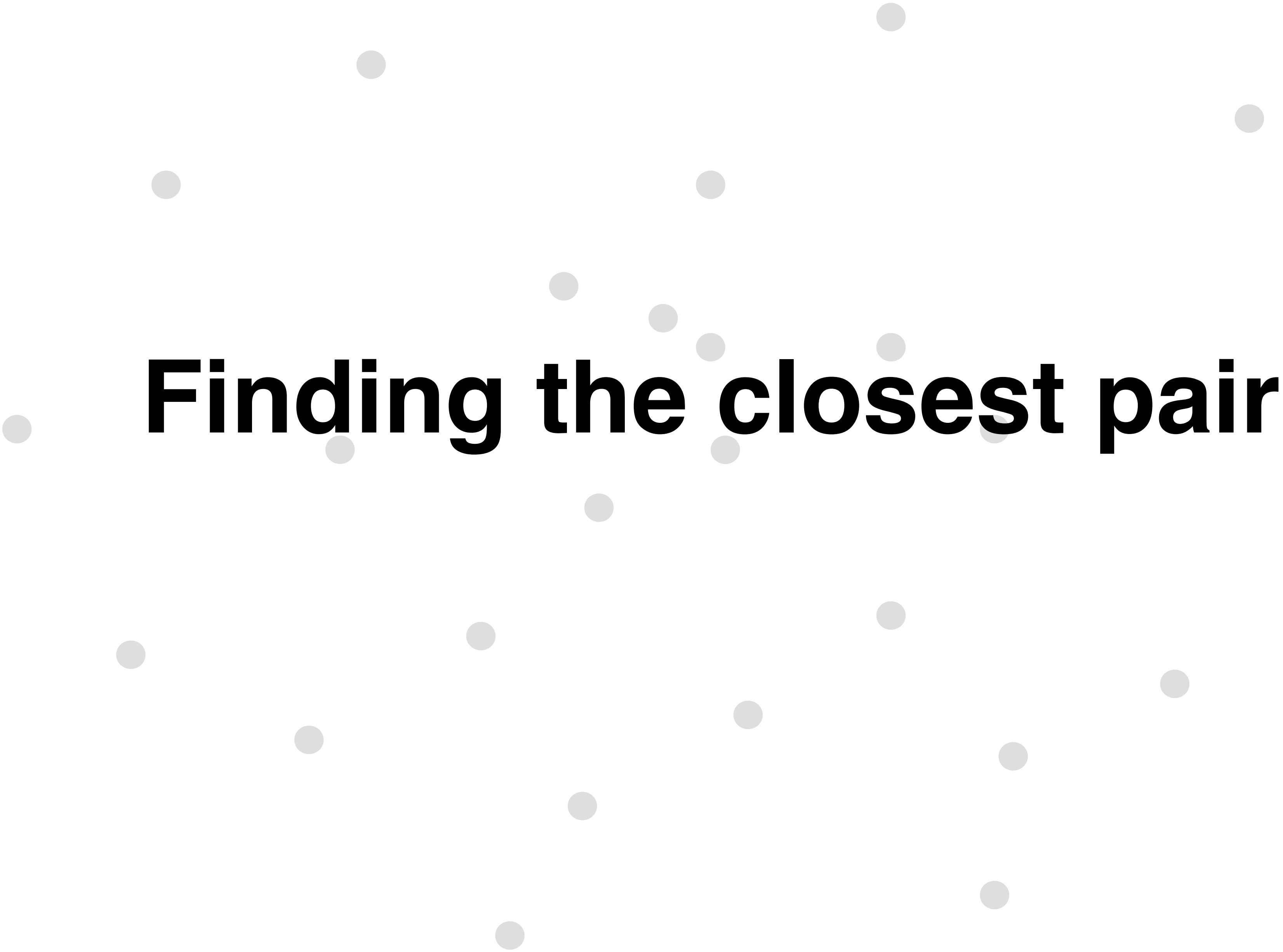
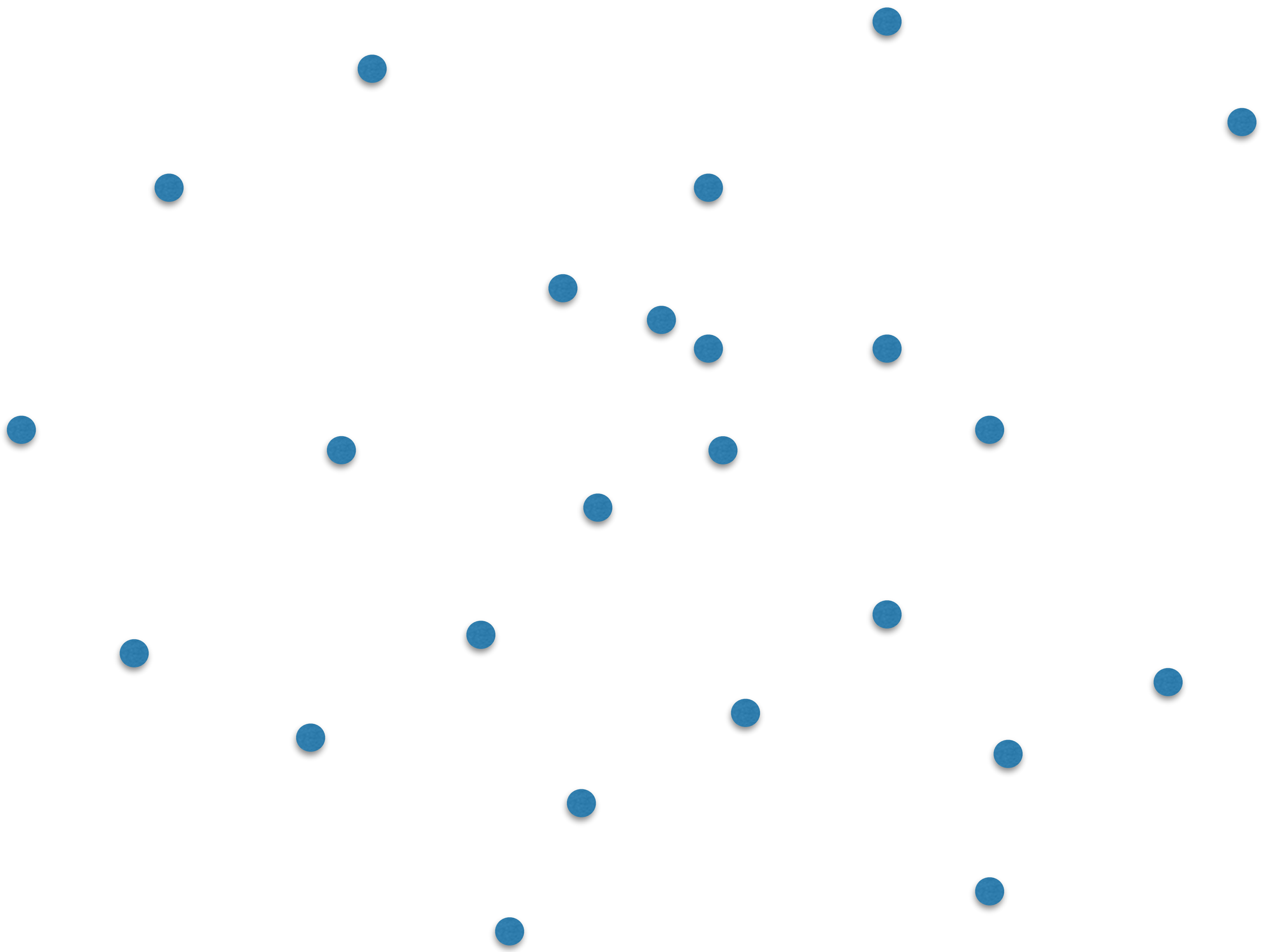


Finding the closest pair

Computational Geometry [csci 3250]
Laura Toma
Bowdoin College

Given an array of points in 2D, find the closest pair.



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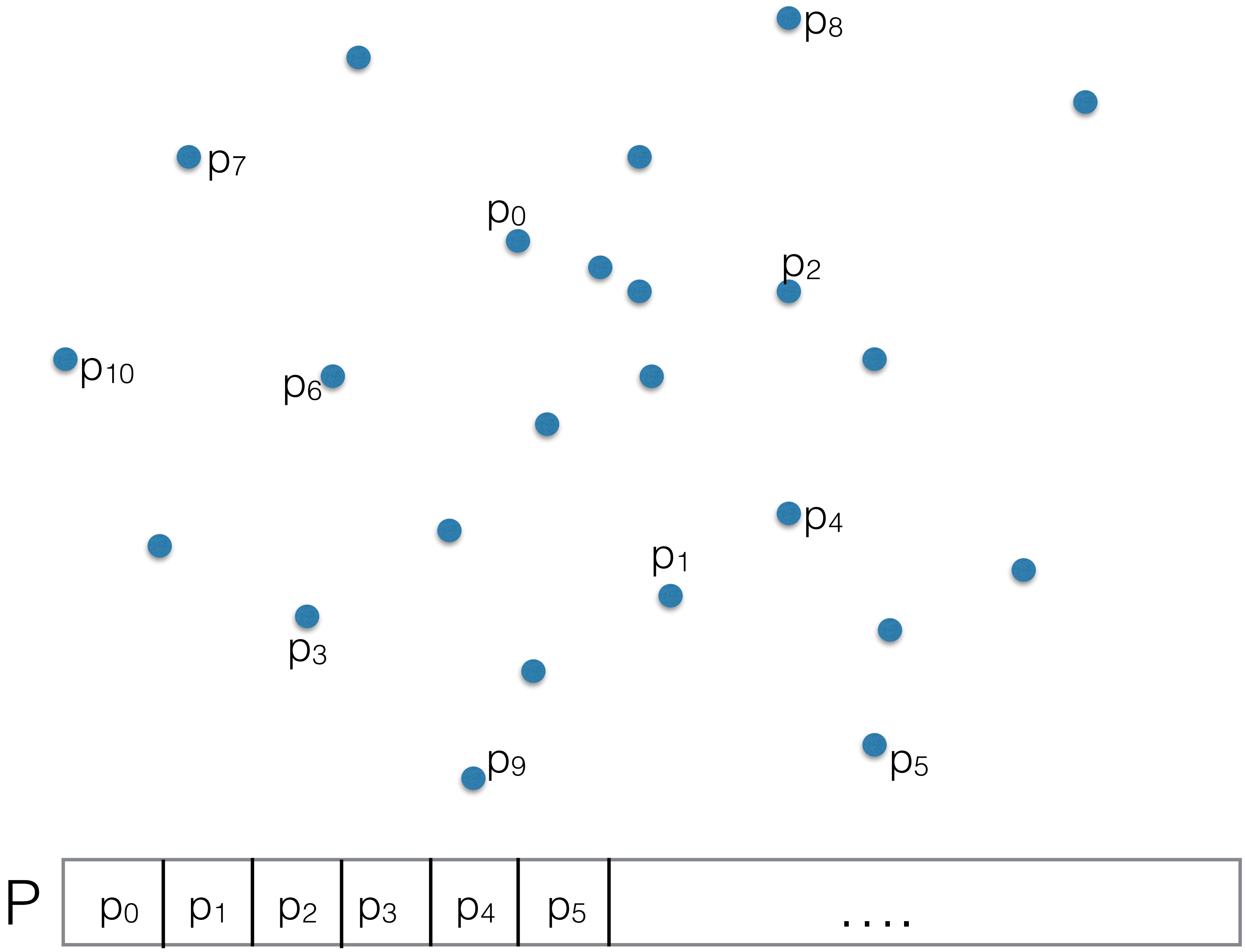

The distance between two points p and q is given by the Euclidian distance given by the formula:

$$d(p,q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$$

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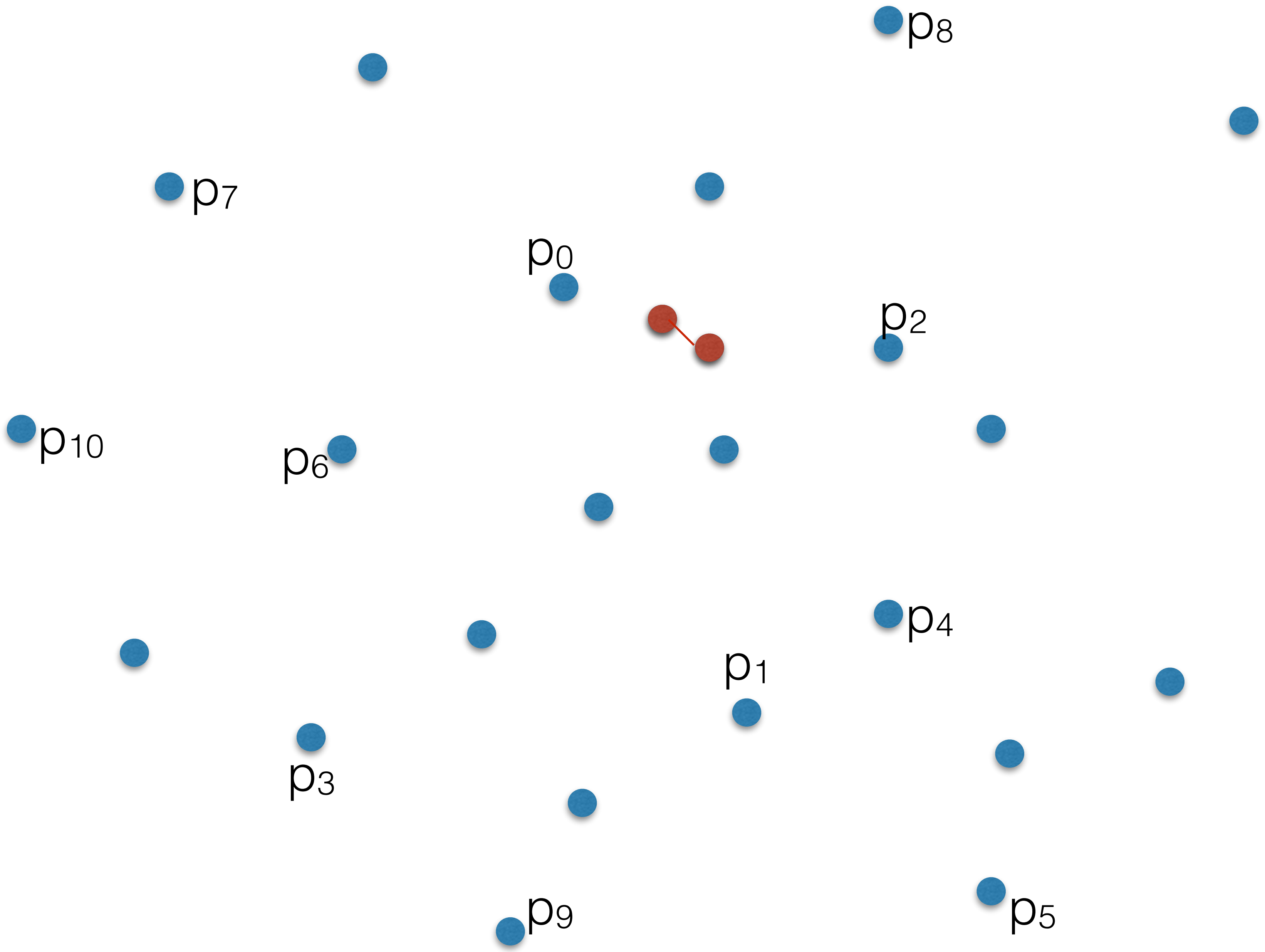
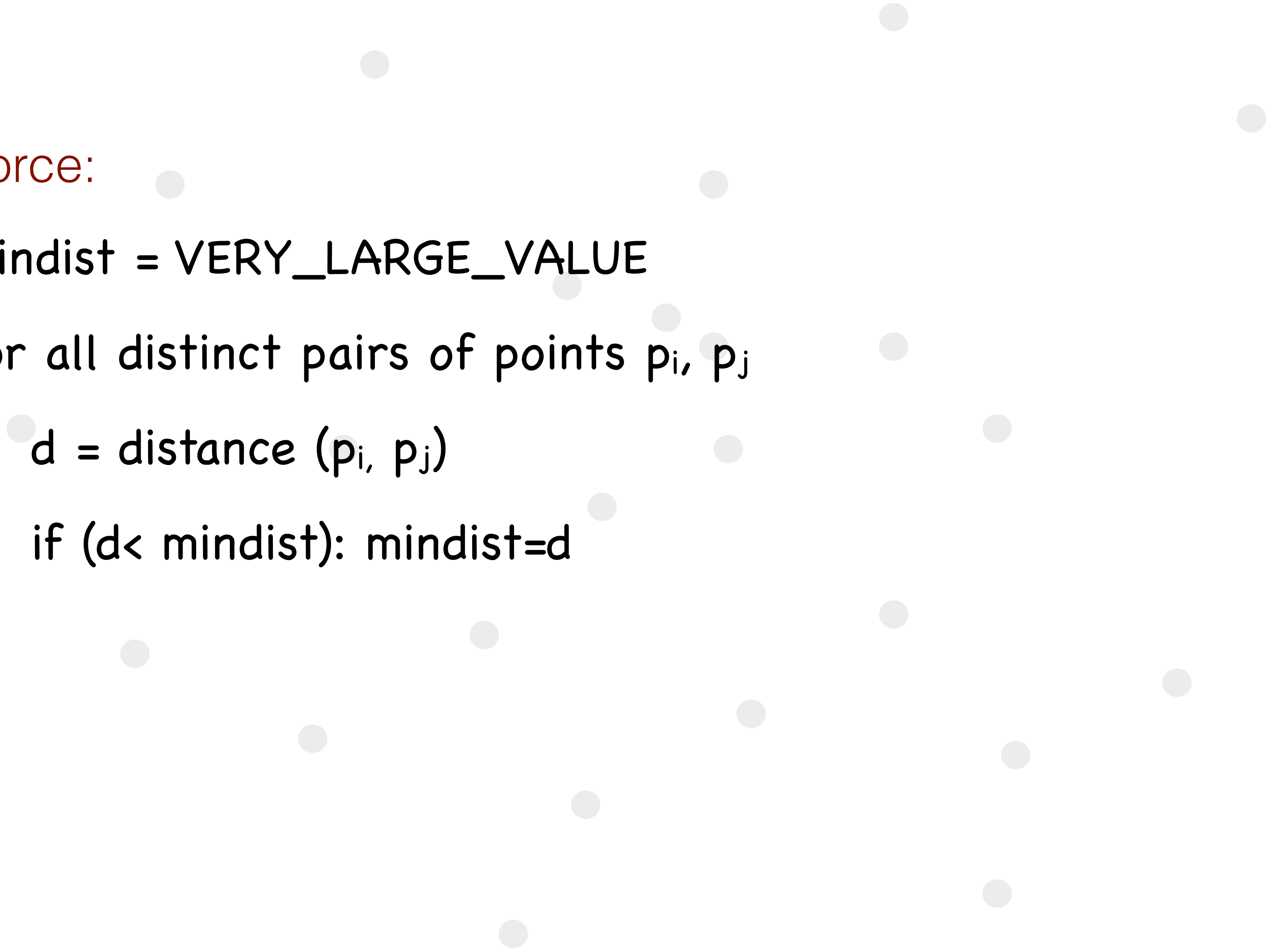


P	p_0	p_1	p_2	p_3	p_4	p_5	. . . .

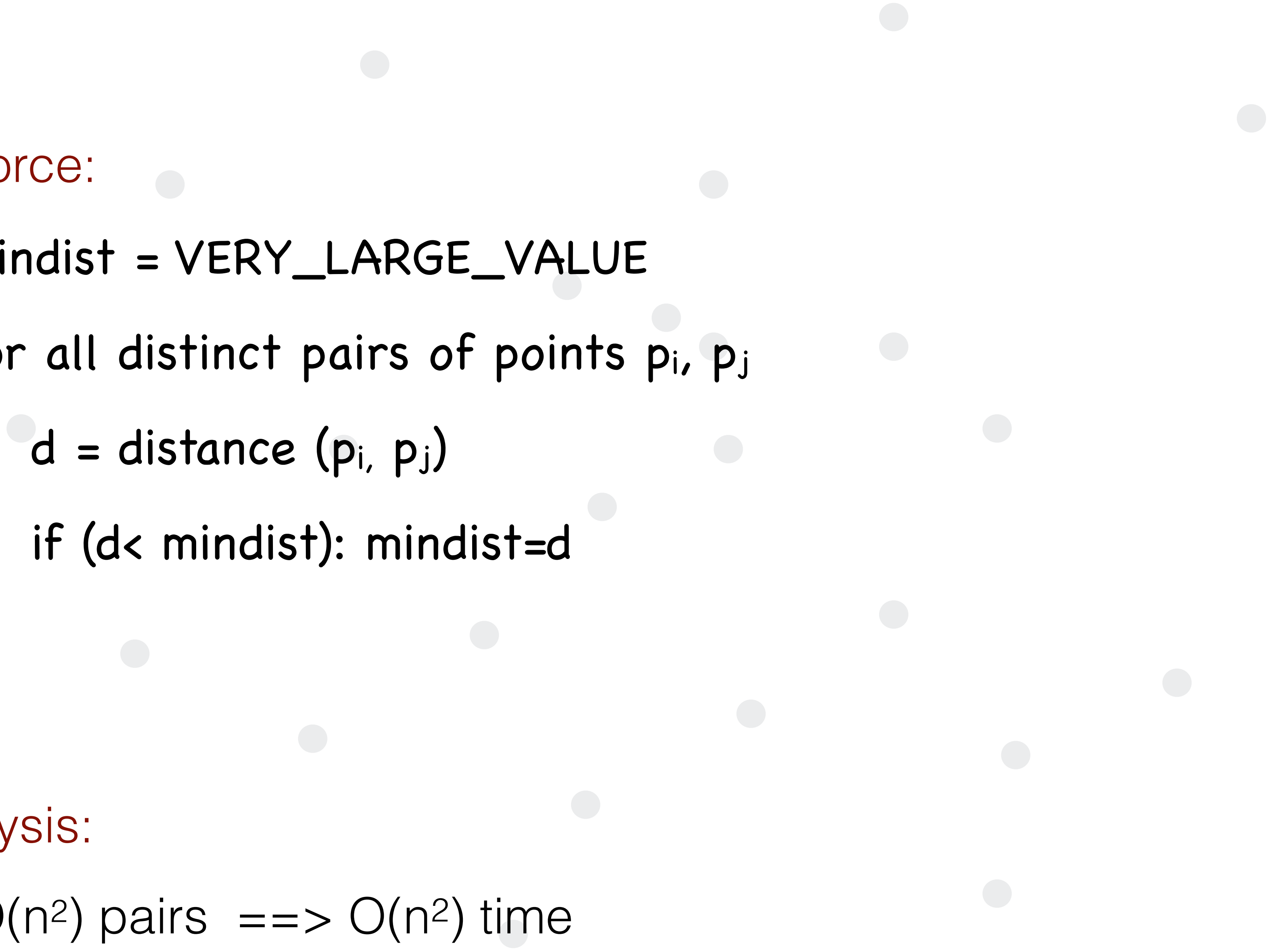
Given an array of points in 2D, find the closest pair.



Brute force:

```
• mindist = VERY_LARGE_VALUE
• for all distinct pairs of points $p_i$, $p_j$
    • d = distance ($p_i$, $p_j$)
    • if (d< mindist): mindist=d
```

Given an array of points in 2D, find the closest pair.



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- Analysis:
  - O(n^2) pairs ==> O(n^2) time

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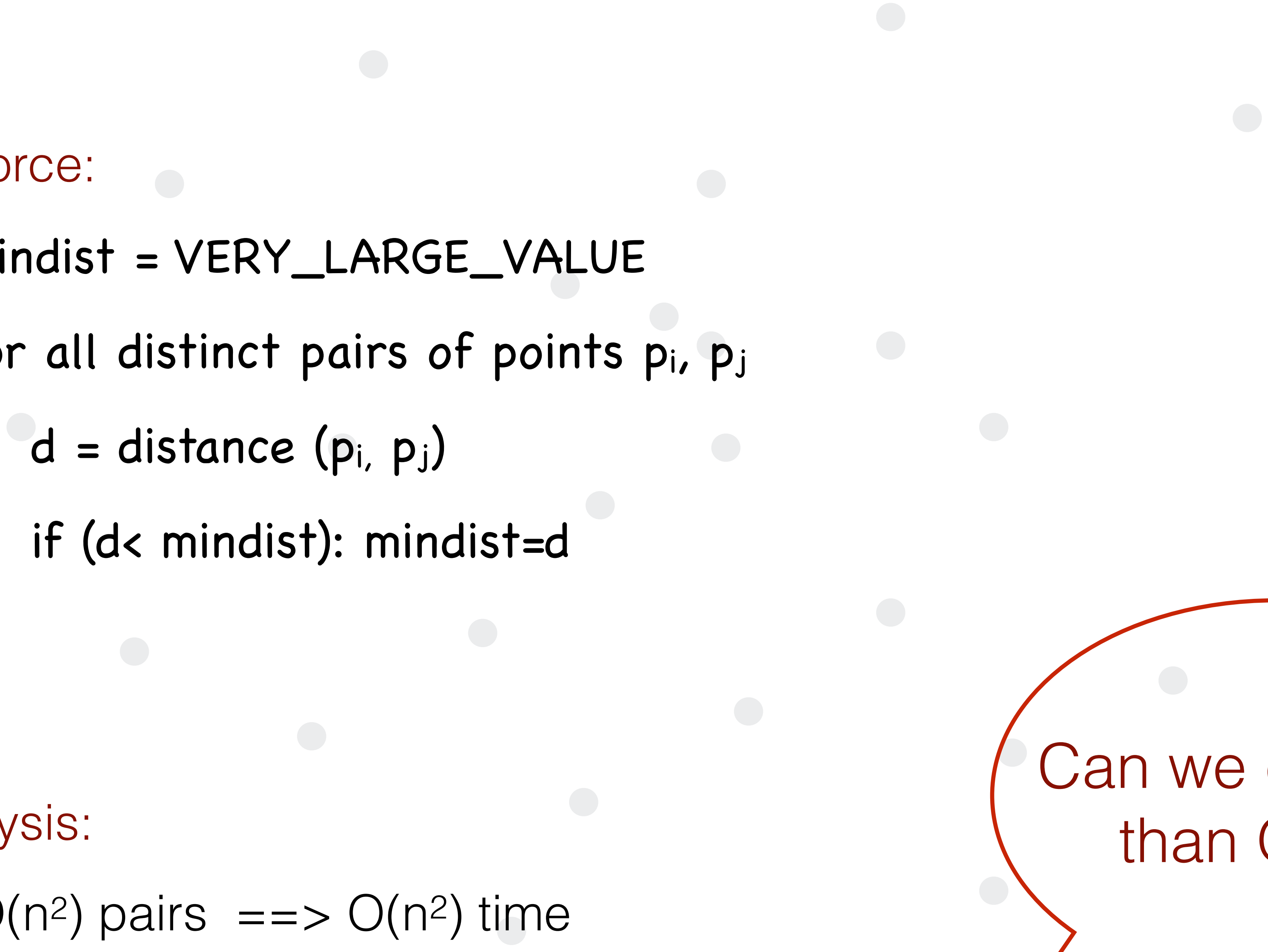

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```

- Analysis:
  - $O(n^2)$ pairs ==> $O(n^2)$ time

Can we do better than $O(n^2)$?

Given an array of points in 2D, find the closest pair.

Can we do better than $O(n^2)$?

Given an array of points in 2D, find the closest pair.

Can we do better than $O(n^2)$?

Hint: use divide-and-conquer

Divide-and-conquer refresher

Divide-and-conquer

```
mergesort(array A)
  • if A has 1 element, there's nothing to sort, so just return it
  • else
      //divide input A into two halves, A1 and A2
      • A1 = first half of A
      • A2 = second half of A
      //sort recursively each half
      • sorted_A1 = mergesort(array A1)
      • sorted_A2 = mergesort(array A2)
      //merge
      • result = merge_sorted_arrays(sorted_A1, sorted_A2)
      • return result
```

Divide-and-conquer

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      • return result
```

Analysis: $T(n) = 2T(n/2) + O(n)$ => $O(n \lg n)$

D&C, in general

```
DC(input P)
  if P is small, solve and return
  else
    //divide
    divide input P into two halves, P1 and P2
    //recurse
    result1 = DC(P1)
    result2 = DC(P2)
    //merge
    do_something_to_figure_out_result_for_P

    return result
```

Analysis: $T(n) = 2T(n/2) + O(\text{merge phase})$

D&C, in general

```
DC(input P)
  if P is small, solve and return
  else
    //divide
    divide input P into two halves, P1 and P2
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    result1 = DC(P1)
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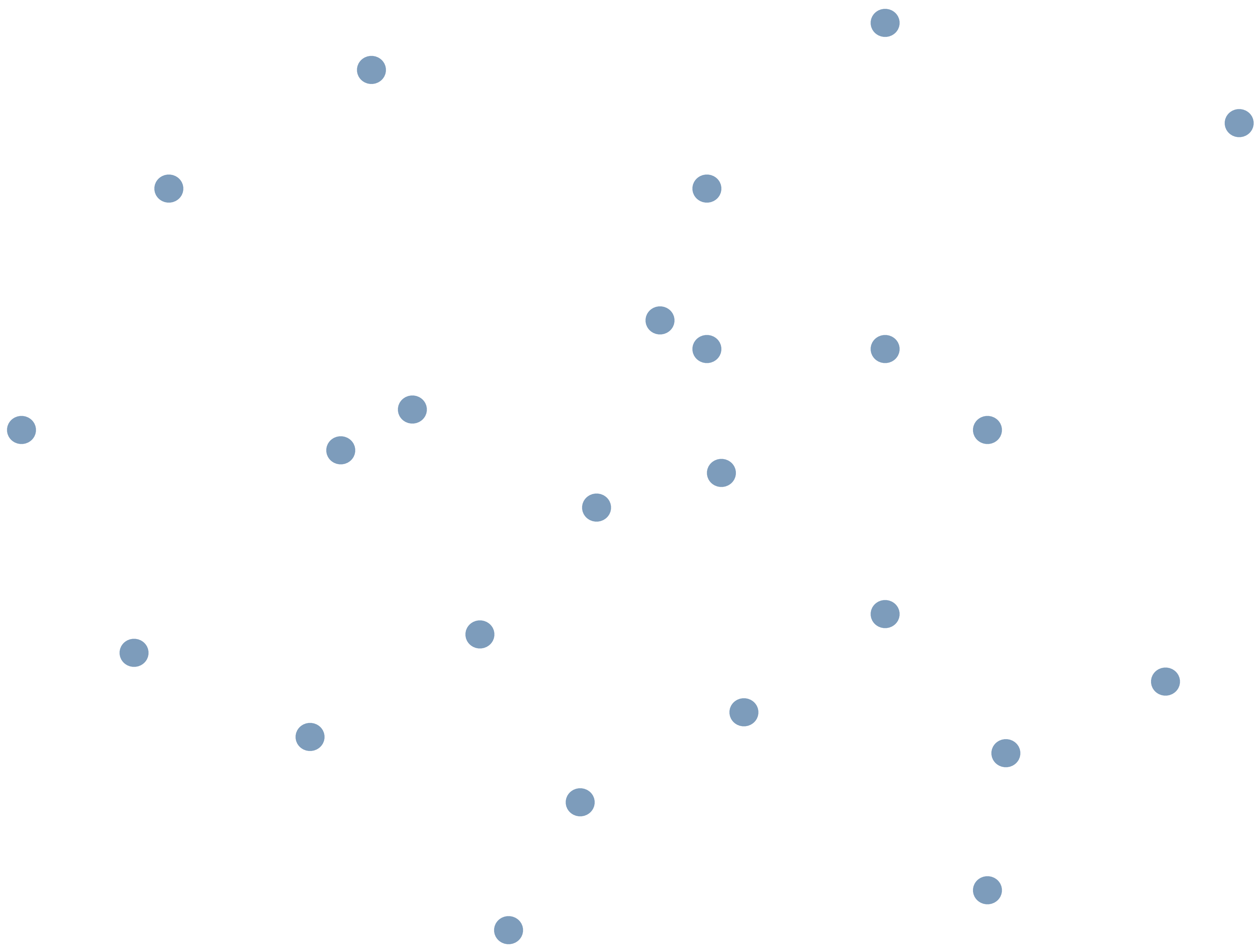
    //merge
    do_something_to_figure_out_result_for_P

    return result
```

Analysis: $T(n) = 2T(n/2) + O(\text{merge phase})$

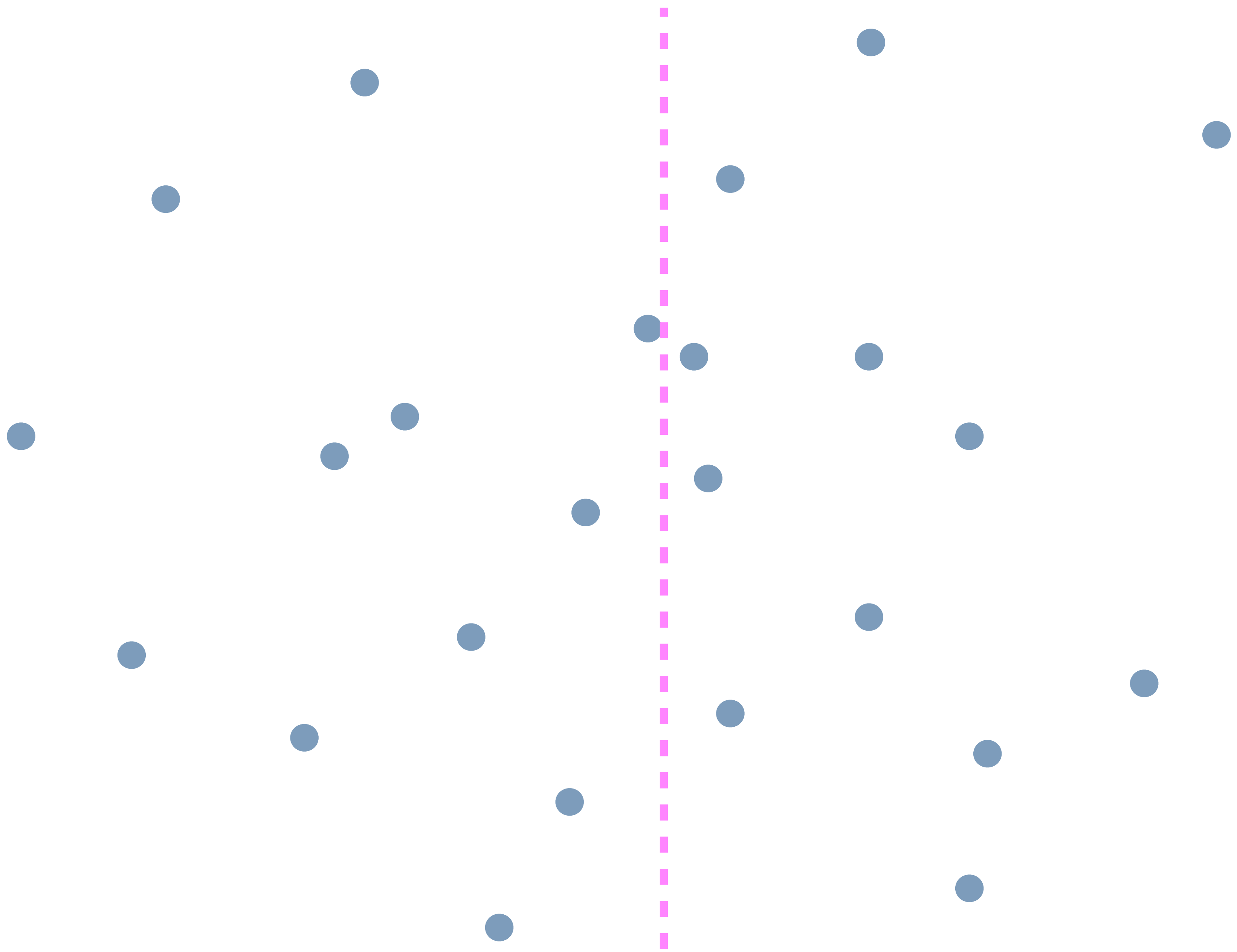
- if merge phase is **O(n):** $T(n) = 2T(n/2) + \mathbf{O(n)}$ => $O(n \lg n)$
- if merge phase is **O(n lg n)**: $T(n) = 2T(n/2) + \mathbf{O(n \lg n)}$ => $O(n \lg^2 n)$

Closest pair, divide-and-conquer



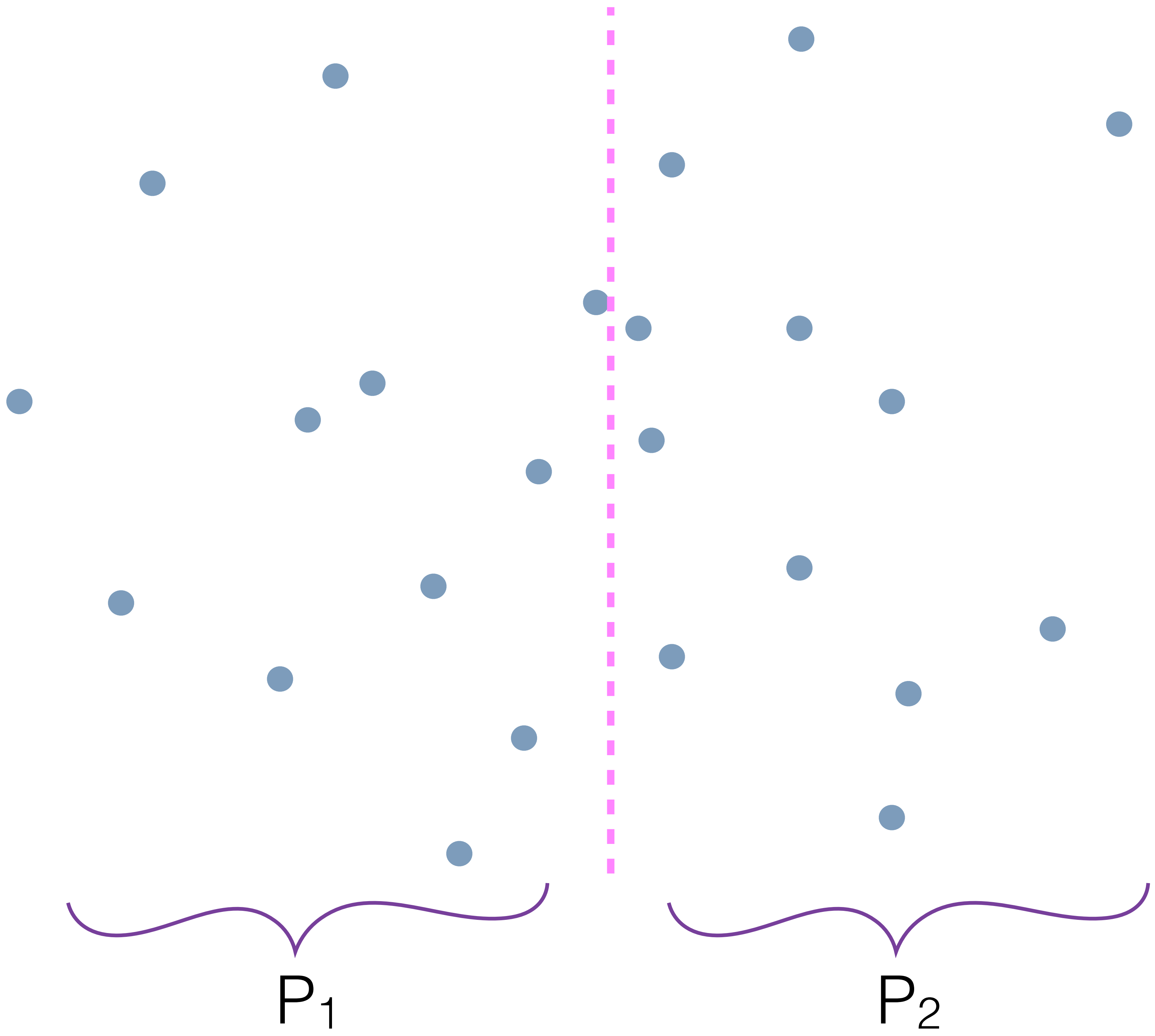
Closest pair, divide-and-conquer

- find vertical line that splits P in half



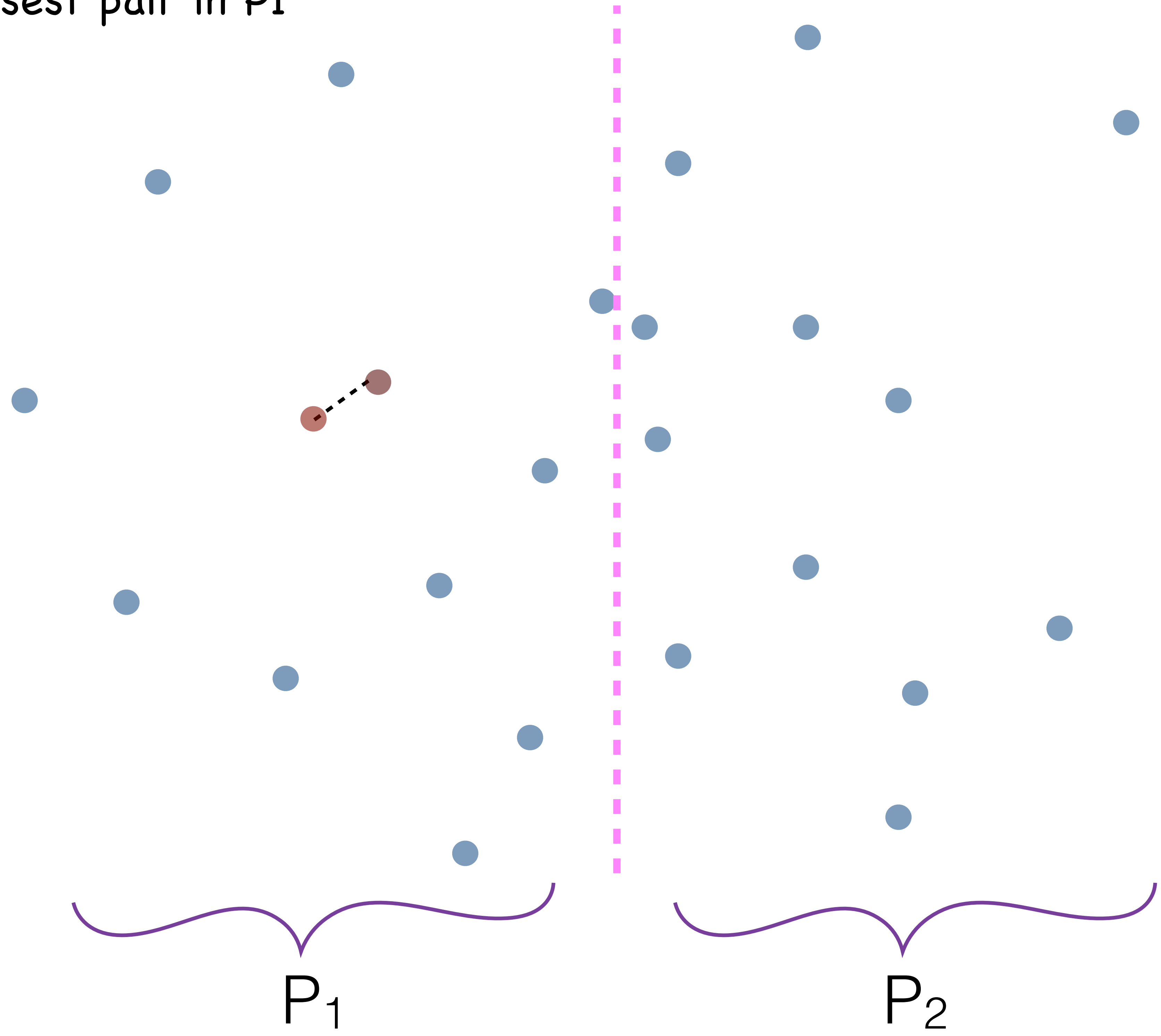
Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line



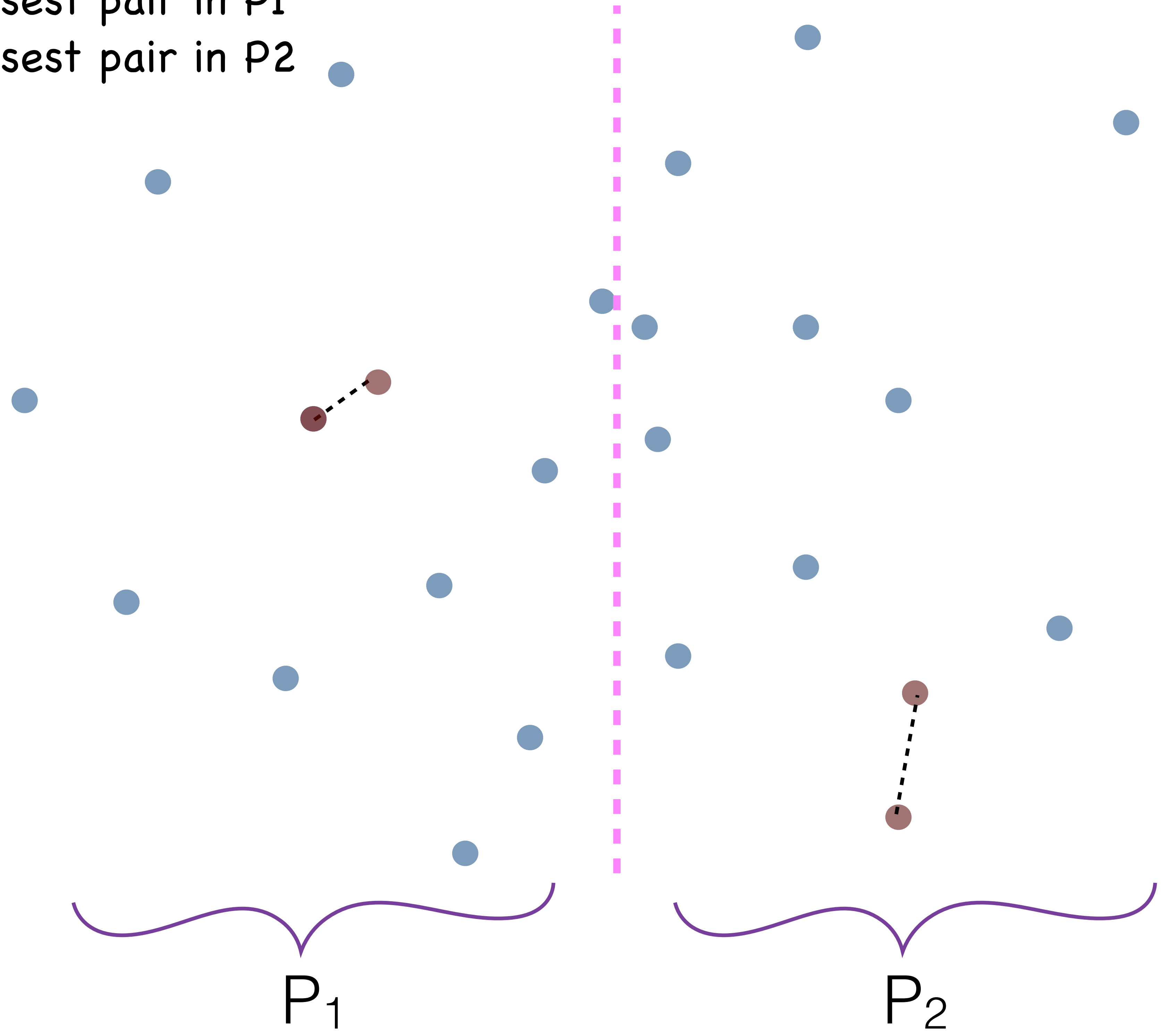
Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1



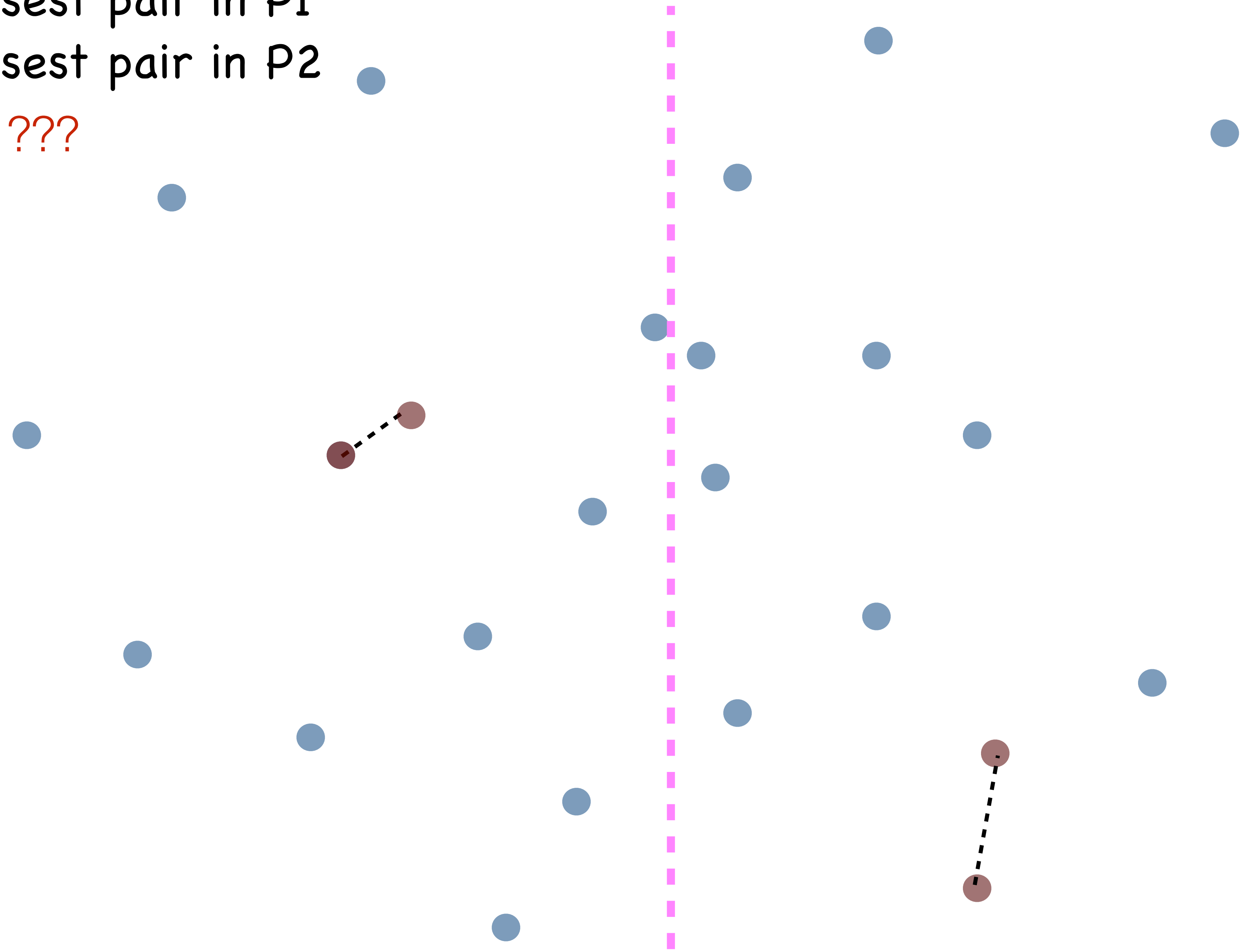
Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
- recursively find closest pair in P2



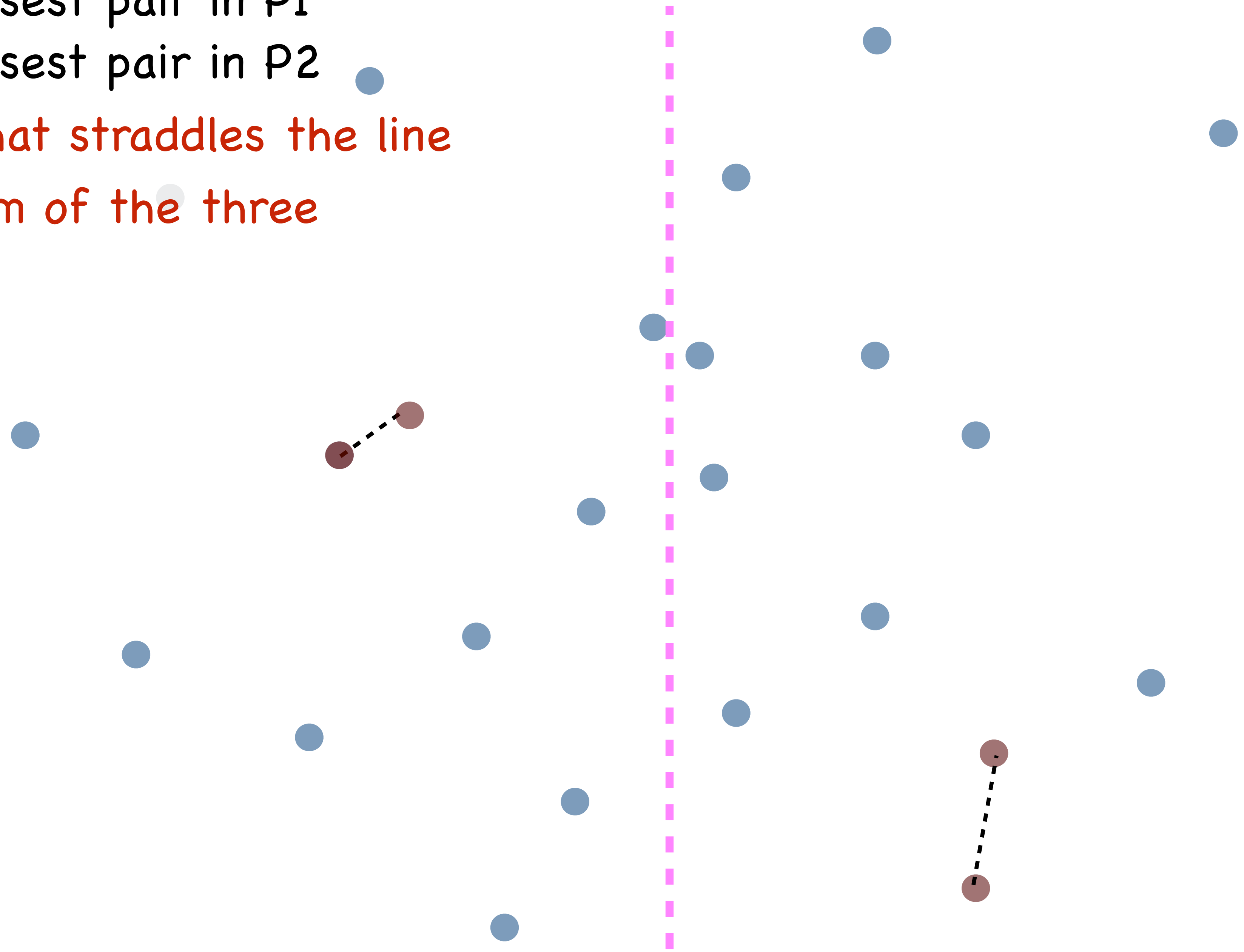
Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
- recursively find closest pair in P2
- //...... NOW WHAT ???



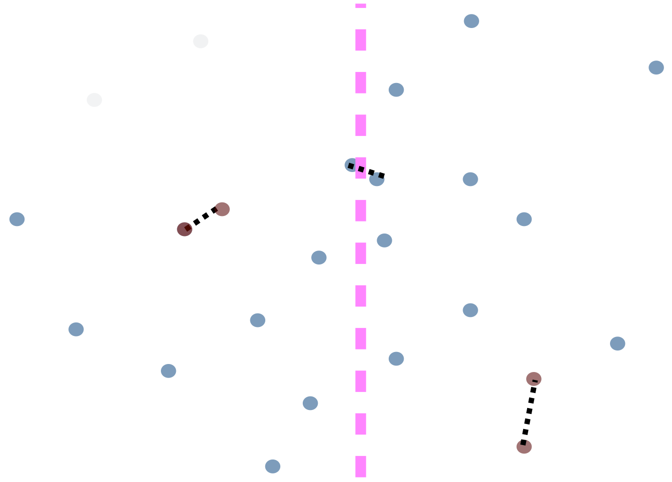
Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
- recursively find closest pair in P2
- find closest pair that straddles the line
- return the minimum of the three



Closest pair, divide-and-conquer

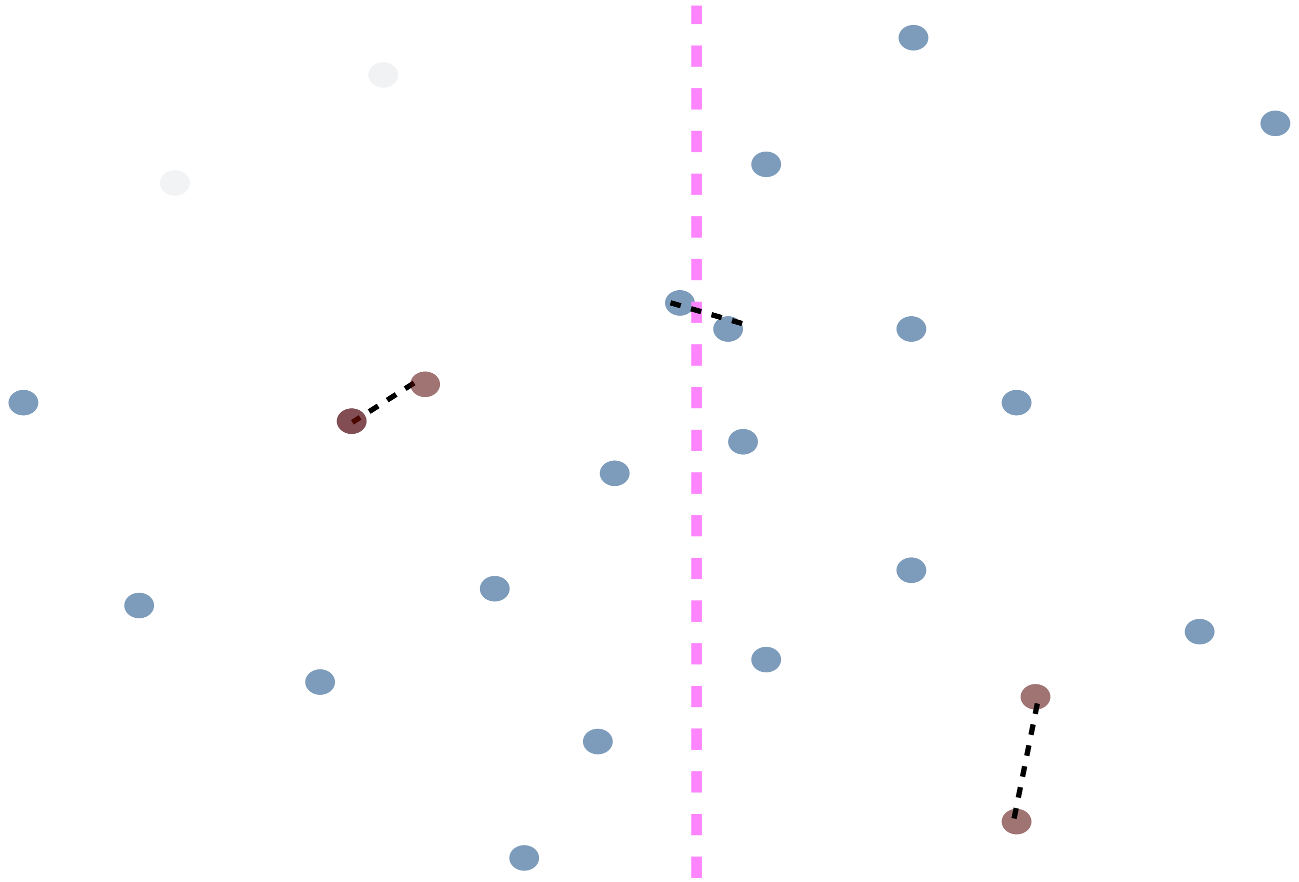
```
FindClosestPair(P)
  //basecase
  • if P has 1 point, return infinity
  • if P has 2 points, return their distance
  • else
      • find vertical line that splits P in half
      • let P1, P2 = set of points to the left/right of line
      • $d_1$ = FindClosestPair(P1)
      • $d_2$ = FindClosestPair(P2)
      //compute closest pair across
      • mindist=infinity
      • for each p in $P_1$, for each q in $P_2$
          • compute distance d(p,q)
          • mindist = min{$d_1$, $d_2$, d(p,q)}
      //return smallest of the three
      • return min {$d_1$, $d_2$, mindist}
```



1. Is this correct?

2. Running time?

Is it correct?



The closest pair in P falls in one of three cases:

- **Both points are in P1:** then it is found by the recursive call on P1
- **Both points are in P2:** then it is found by the recursive call on P2
- **One point is in P1 and one in P2:** then it is found in the merge phase, because the merge phase considers all such pairs

```
FindClosestPair(P)
    //basecase
    • if P has 1 point, return infinity
    • if P has 2 points, return their distance
    • else
        • find vertical line that splits P in half
        • let P1, P2 = set of points to the left/right of line
        • $d_1$ = FindClosestPair(P1)
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        //compute closest pair across
        • mindist=infinity
        • for each p in $P_1$, for each q in $P_2$
            • compute distance d(p,q)
            • mindist = min{$d_1$, $d_2$, d(p,q)}
        //return smallest of the three
        • return min {$d_1$, $d_2$, mindist}
```

Running time?

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    //basecase
    • if P has 1 point, return infinity
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    • else
        • find vertical line that splits P in half
        • let P1, P2 = set of points to the left/right of line
        • $d_1$ = FindClosestPair(P1)
        • $d_2$ = FindClosestPair(P2)
        //compute closest pair across
        • mindist=infinity
        • for each p in $P_1$, for each q in $P_2$
            • compute distance d(p,q)
            • mindist = min{$d_1$, $d_2$, d(p,q)}
        //return smallest of the three
        • return min {$d_1$, $d_2$, mindist}
```

Running time?

$T(n) = 2T(n/2) + O(n^2)$

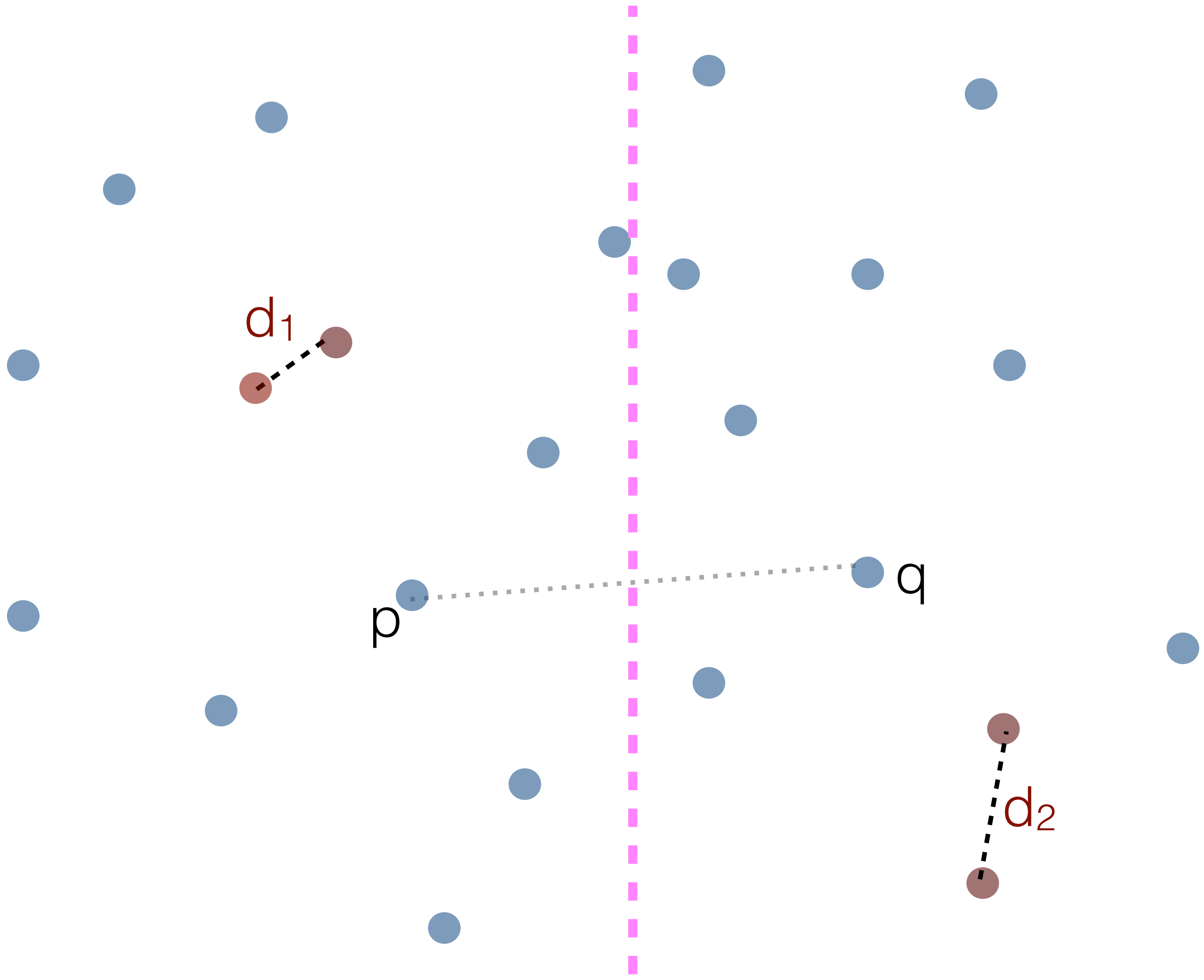
solves to $O(n^2)$

Can we do better?

Refining the merge

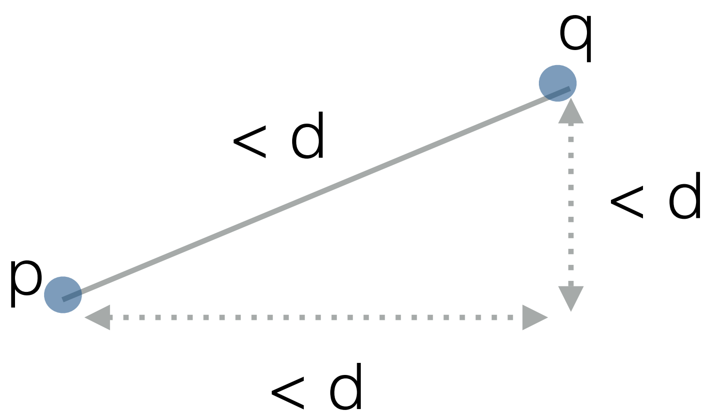
Do we need to examine all pairs {p,q}, with p in P_1, q in P_2?

Which pairs {p,q} can be discarded?

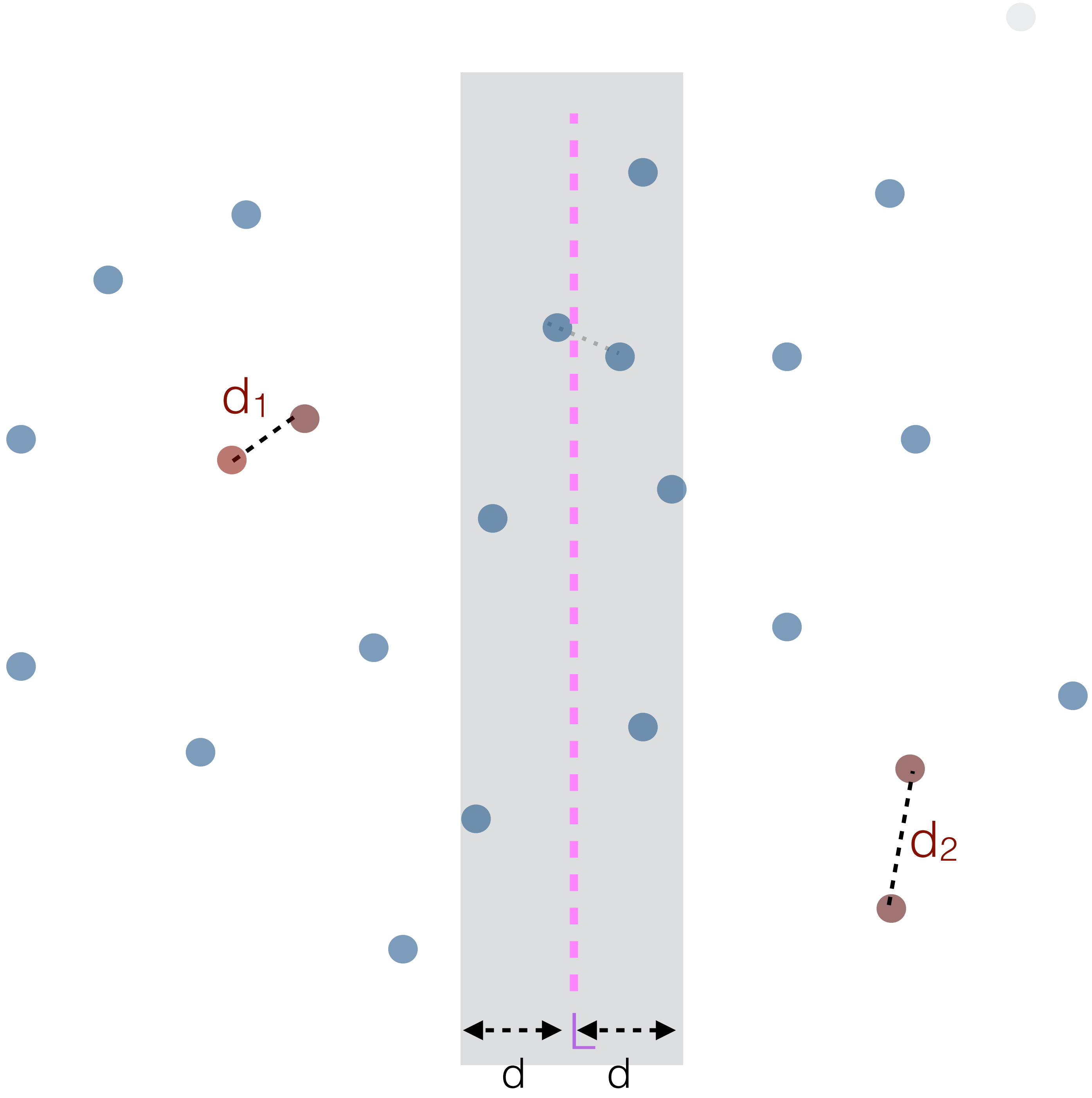


Here’s a very simple observation..

- Notation: $d = \min\{d_1, d_2\}$
- Observation: If there is a pair of points $\{p,q\}$ with $\text{dist}(p,q) < d$, then both the horizontal and vertical distance between p and q must be smaller than d.

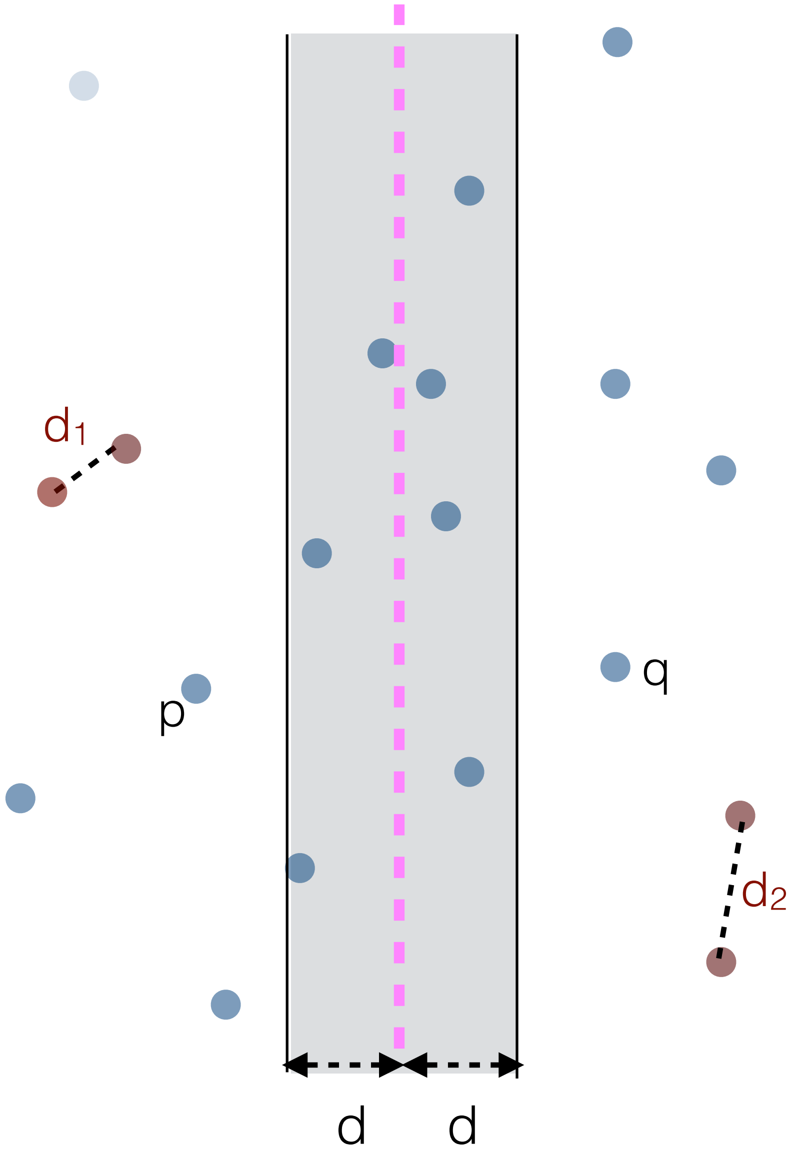


- Notation: $d = \min \{d_1, d_2\}$
- Furthermore, if there is a pair of points $\{p,q\}$ with $\mathrm{dist}(p,q) < d$, then both p and q must be within distance d from line L.



Refining the merge

```
FindClosestPair(P)
  • if P has 1 point, return infinity
  • if P has 2 points, return their distance
  • else
      • find vertical line that splits P in half
      • let P1, P2 = set of points to the left/right of line
      • d1=FindClosestPair(P1)
      • d2=FindClosetPair(P2)
      • traverse $P_1$ and select all points $P_1'$ in the strip
      • traverse $P_2$ and select all points $P_2'$ in the strip
      • for each p in $P_1'$
          • for each point q in $P_2'$
              • compute distance d(p,q)
              • mindist = min{$d_1$, $d_2$, d(p,q)}
  • return mindist
```



Running time?

Refining the merge

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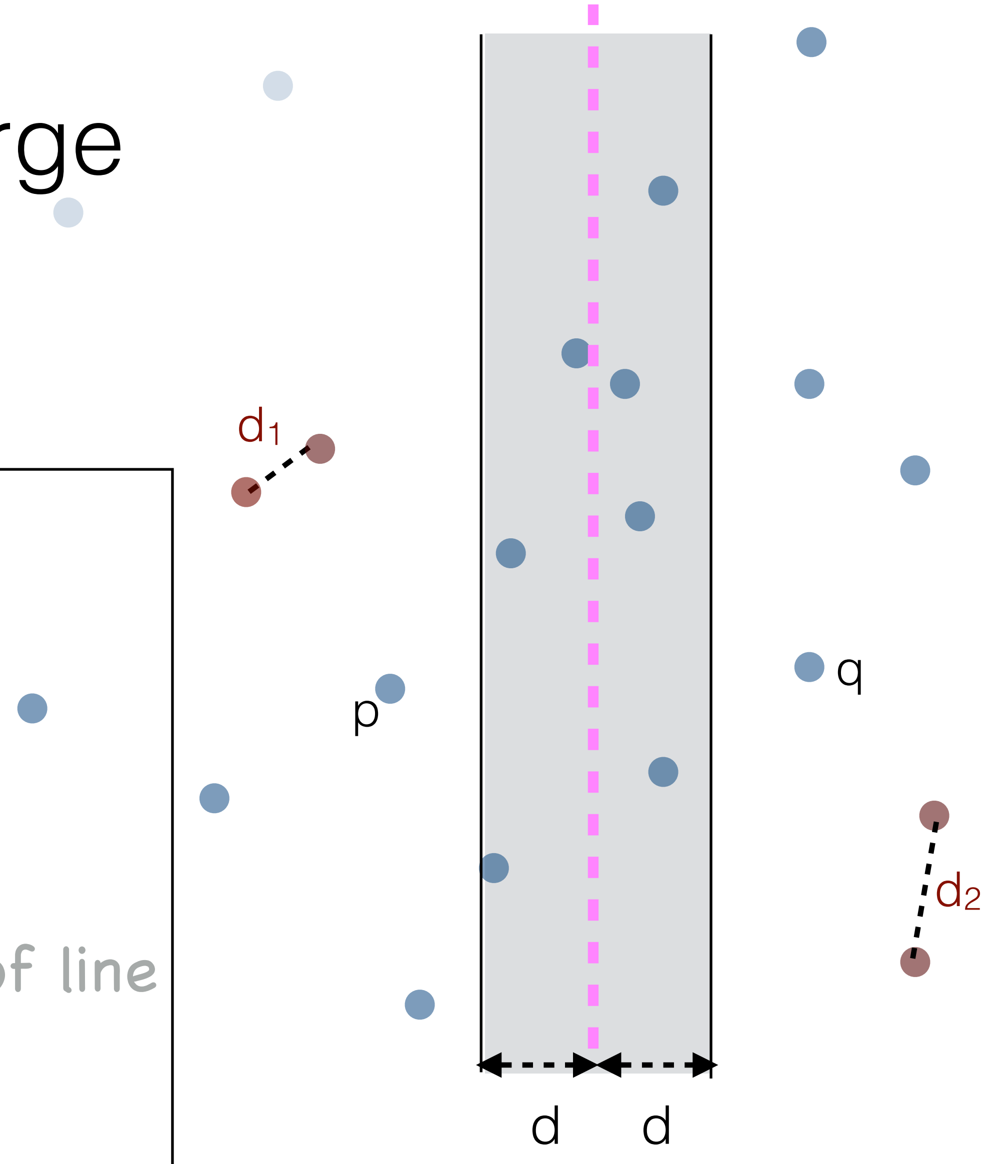
      • let P1, P2 = set of points to the left/right of line
      • d1=FindClosestPair(P1)
      • d2=FindClosetPair(P2)
      • traverse $P_1$ and select all points $P_1'$ in the strip
      • traverse $P_2$ and select all points $P_2'$ in the strip
      • for each p in $P_1'$
          • for each point q in $P_2'$
              • compute distance d(p,q)
              • mindist = min{$d_1$, $d_2$, d(p,q)}
  • return min {$d_1$, $d_2$, mindist}
```

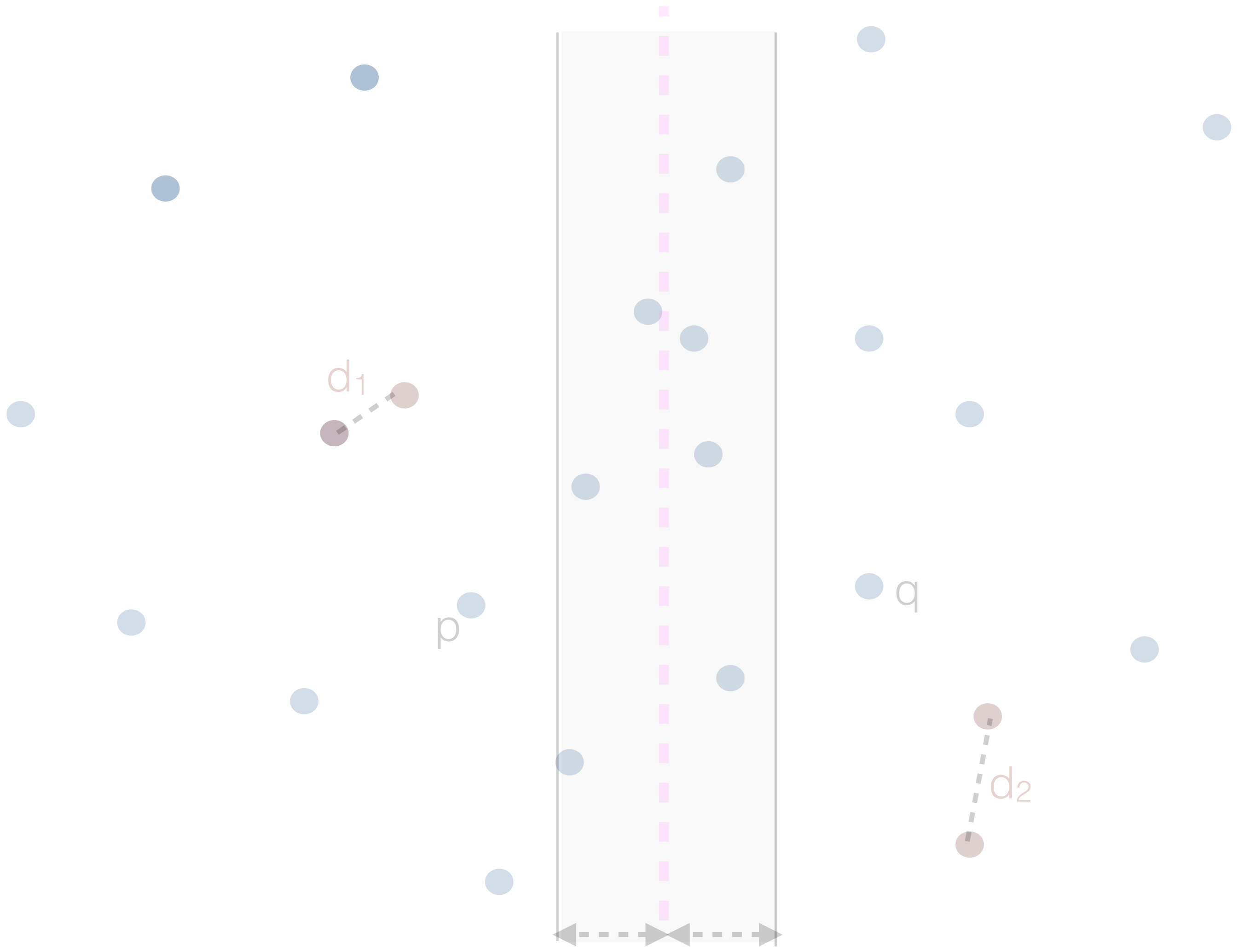


Running time?

It's possible that all n/2 points on either side lie inside the strip

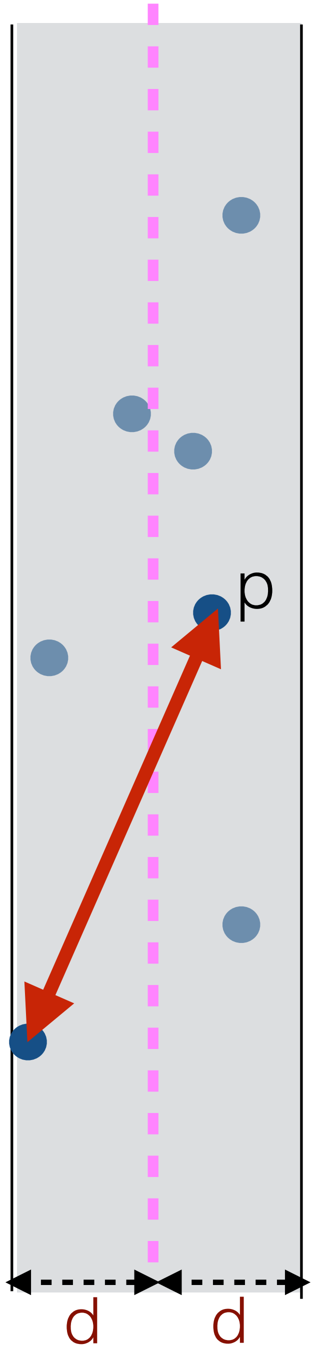
Refining the merge

- Show an example where the strip may contain Omega(n) points.
- What does this imply for the running time?



Refining the merge

- Filtering the points in the strip is not enough..
- Note that the strip contains candidate pairs that could be within distance d of each other horizontally
- We haven't used yet that candidate pairs have to be within distance d of each other vertically



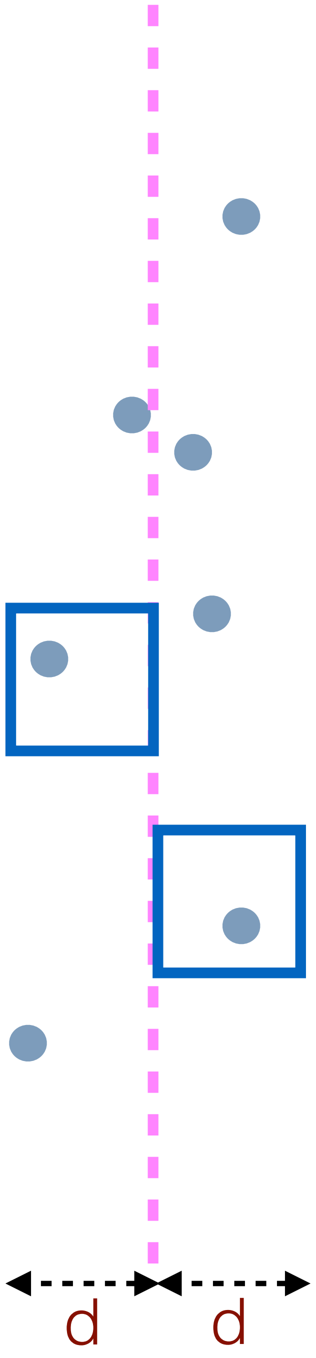
{p,q} not a candidate pair because their vertical distance > d

Refining the merge

Notation: $d = \min \{d_1, d_2\}$

Not only candidate pairs must be in the d-by-d strip around line L, but….

Points on both sides are “sparse”



Any pair of points in P_1 must be at least d away

Any pair of points in P_2 must be at least d away

Refining the merge

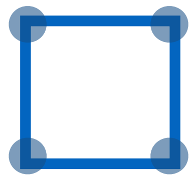
Notation: $d = \min\{d_1, d_2\}$

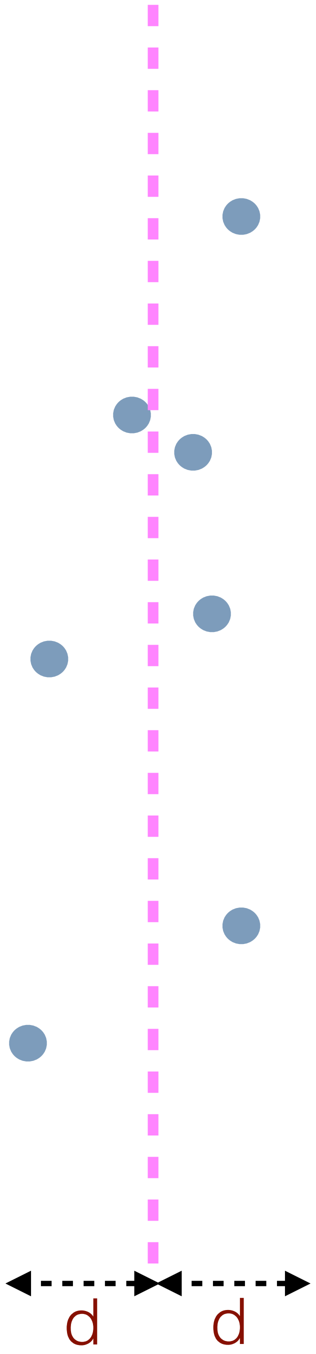
Not only candidate pairs must be in the d-by-d strip around line L, but….

Points on both sides are “sparse”

Any pair of points in P_1 must be at least d away

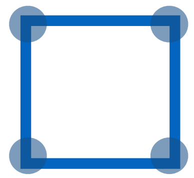
Any square with side d contains at most 4 points of P_1





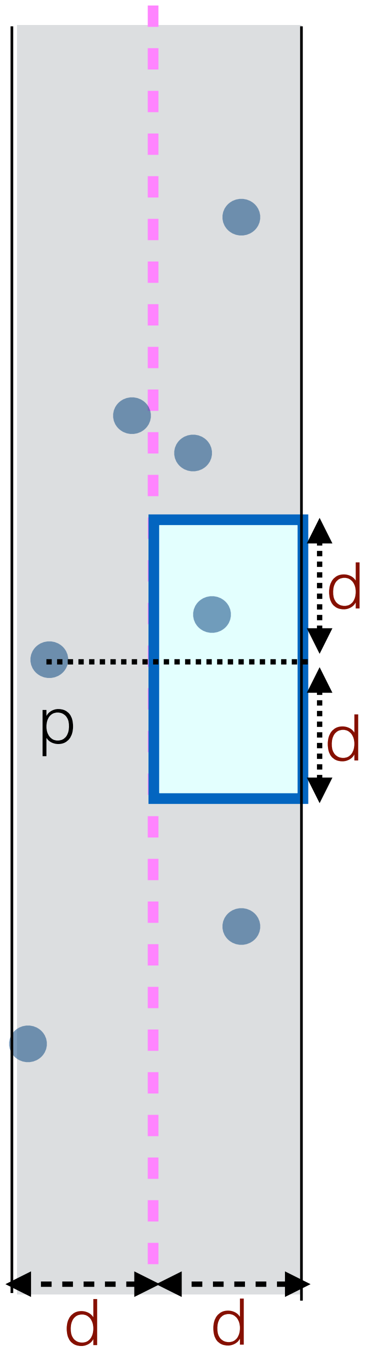
Any pair of points in P_2 must be at least d away

Any square with side d contains at most 4 points of P_2



How can we use this?

- Consider a point p in P_1'
- We don't need to compute the distances from p to all points in P_2'



- All points of P_2' within distance d of p are vertically above or below p by at most d
- => they must lie in a rectangle of size d x 2d

How can we use this?

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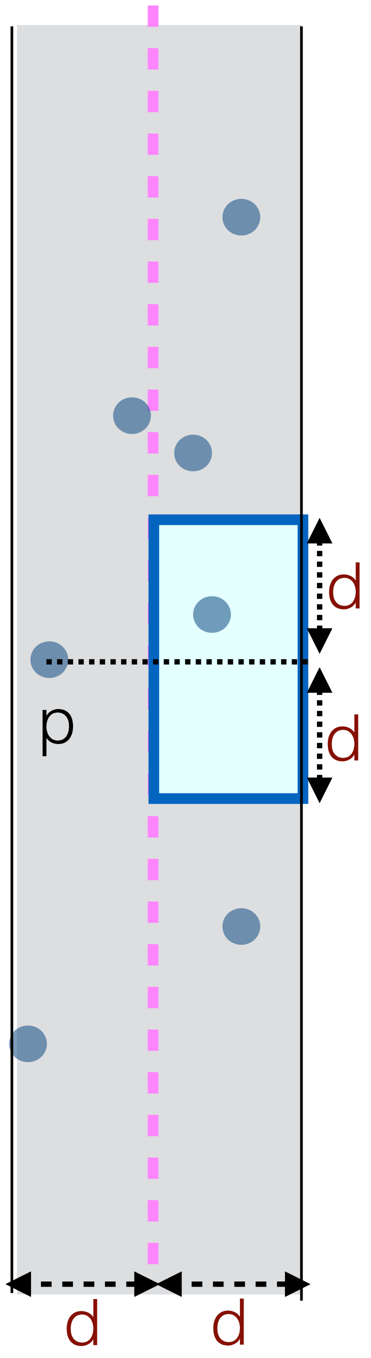
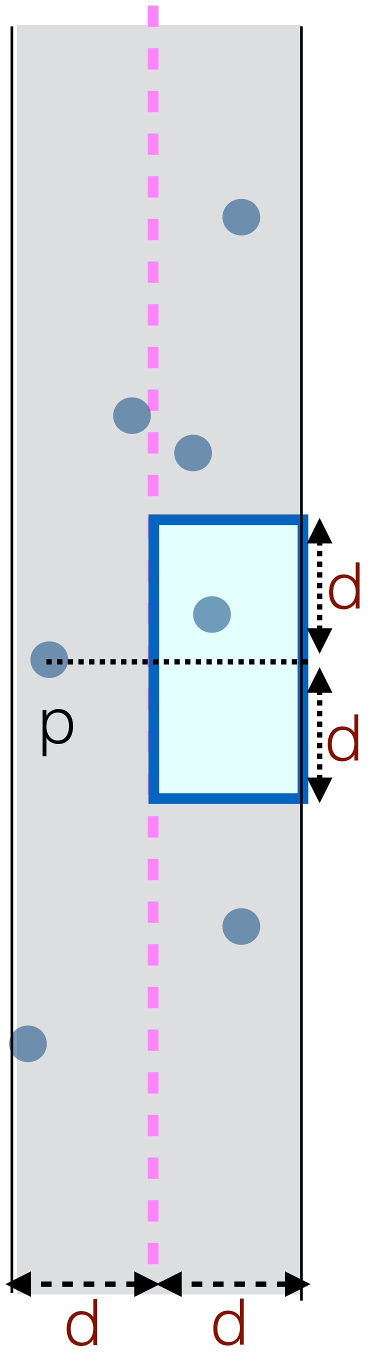


- All points of P_2' within distance d of p are vertically above or below p by at most d
- => they must lie in a rectangle of size d x 2d

- How many points q of P_2' can there be in a rectangle of size d x 2d? (knowing that any pair of points in P_2' must be at least d away).

How can we use this?

- Consider a point p in P_1'
- We don't need to compute the distances from p to all points in P_2'



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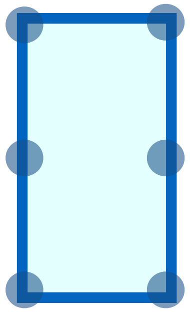
- => they must lie in a rectangle of size d x 2d

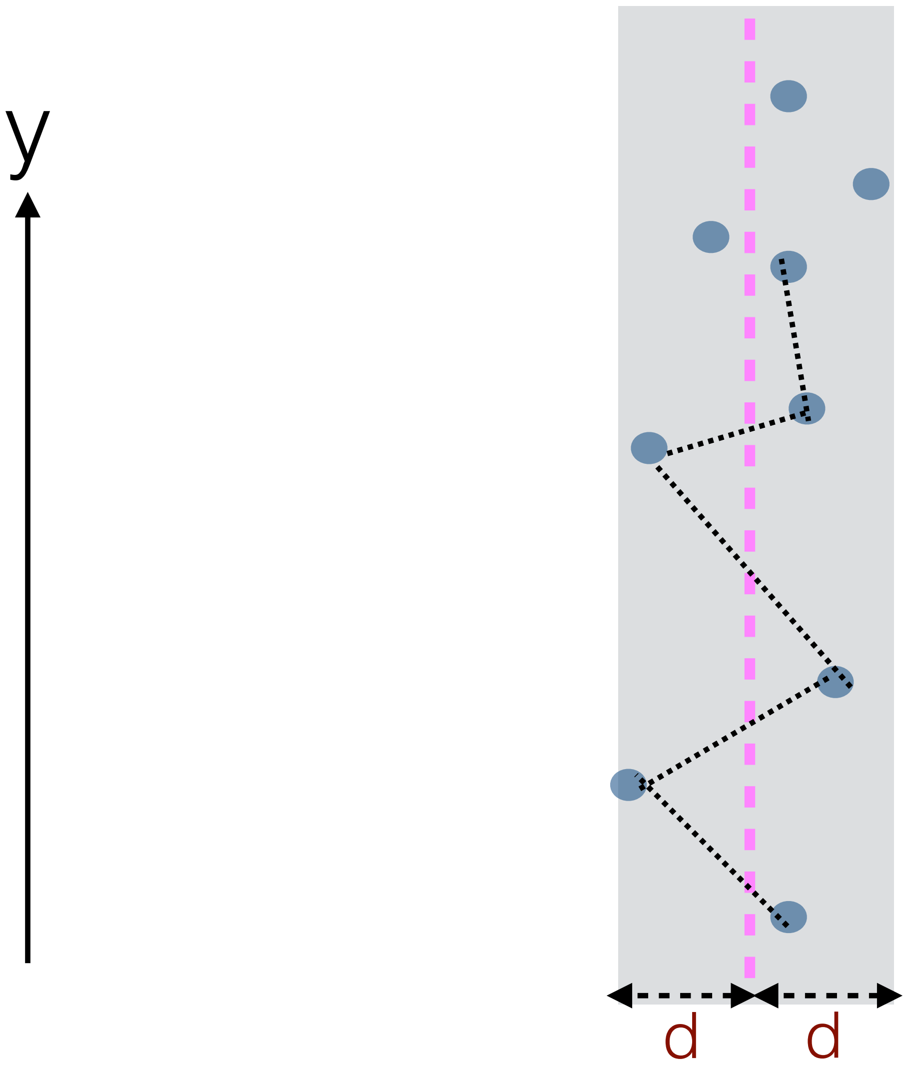
- How many points q of P_2' can there be in a rectangle of size d x 2d? (knowing that any pair of points in P_2' must be at least d away).

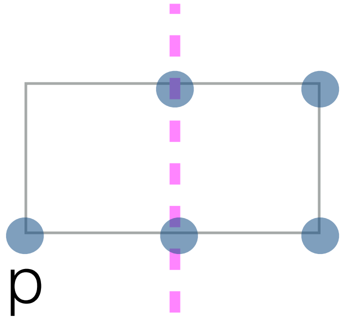


=> So for every p in P_1', we only need to check at most 6 points of P_2'

Refining the merge

- Traverse the points in P_1' and P_2' in increasing order of their y-coordinate
- Mimic the process of merging P_1' and P_2' in y-order
- Consider the next point p in y-order and let's say it comes from P_1'
  - p will check only the points above it (following it in y-order) in P_2'
  - There can be at most 4 subsequent points in P_2' that are within d from p.



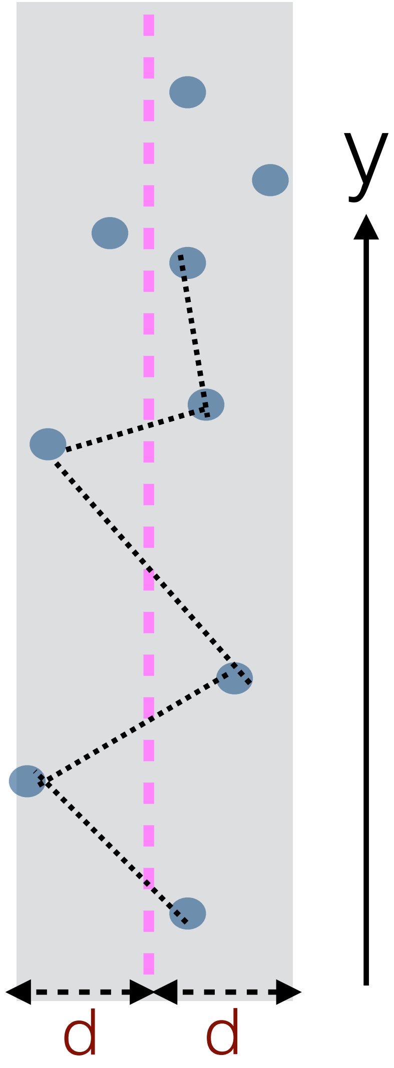


Note: Assume no duplicate points.

Refining the merge

```
closestPair(P)
    //divide
    • find vertical line l that splits P in half
    • let $P_1$, $P_2$ = set of points to the left/right of line
    • $d_1$ = closestPair($P_1$)
    • $d_2$ = closestPair($P_2$)
    //merge
    • let d = min{$d_1$, $d_2$}
    • for all p in $P_1$: if $x_p > x_l - d$: add p to Strip1
    • for all p in $P_2$: if $x_p < x_l + d$: add p to Strip2
    • sort Strip1, Strip2 by y-coord
    • initialize mindist=d
    • merge Strip1, Strip2: for next point p,
        • compute its distance to the 5 points that come after it on the
          other side of the strip
        • if any of these is smaller than mindist, update mindist

    • return mindist
```



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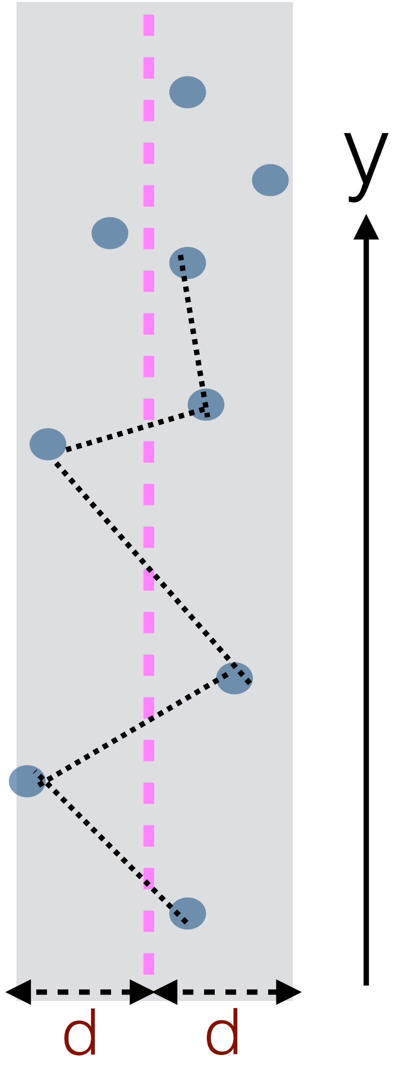
    • initialize mindist=d
    • merge Strip1, Strip2: for next point p,
        • compute its distance to the 5 points that come after it on the other side of the strip
        • if any of these is smaller than mindist, update mindist

    • return mindist
```

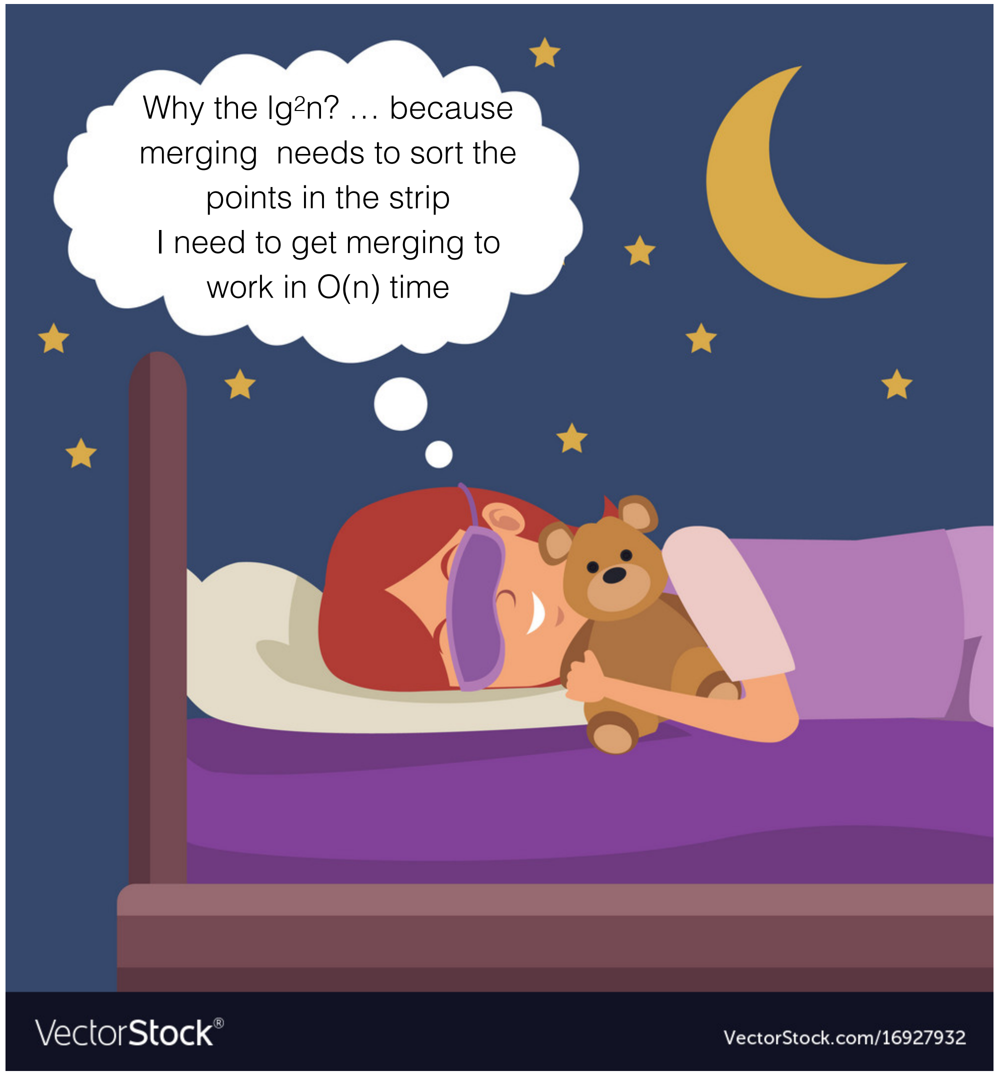


Analysis: $T(n) = 2T(n/2) + O(n \lg n) \Rightarrow O(n \lg^2 n)$

- Brute force: $O(n^2)$
- Divide-and-conquer with smart merge: $O(n \lg^2 n)$

Can we do better?

- We'd love to get rid of the extra lg n



Refining the refined merge

- Instead of sorting inside every merge, …
- Pre-sort P at the beginning
    - sort by x-coord: PX <—— not necessary, but practical
    - sort by y-coord: PY

closestPair(PX, PY)

- Let’s see what that means

Refining the refined merge

closestPair(PX, PY)

```
//divide
  • find vertical line L that splits P in half
  • let $P_1$, $P_2$ = set of points to the left/right of line <— We need to get P1X, P1Y, P2X, P2Y
  • $d_1$ = ~~closestPair($P_1$)~~   closestPair(P1X, P1Y)
  • $d_2$ = ~~closestPair($P_2$)~~   closestPair(P2X, P2Y)
//merge
  • let d = min{$d_1$, $d_2$}
  • ~~for all p in $P_1$: if $x_p > x_l - d$: add p to Strip1~~
  • ~~for all p in $P_2$: if $x_p < x_l + d$: add p to Strip2~~
  • ~~sort Strip1, Strip2 by y-coord~~
  • initialize mindist=d
  • merge Strip1, Strip2: for next point p,
      • compute its distance to the 5 points that come after it on the
        other side of the strip
      • if any of these is smaller than mindist, update mindist

  • return mindist
```

How?

Refining the refined merge

closestPair(PX, PY)

//divide

- find vertical line L that splits P in half
- let P_1, P_2 = set of points to the left/right of line <— We need to get P1X, P1Y, P2X, P2Y
- d_1 = ~~closestPair(P_1)~~ closestPair(P1X, P1Y)
- d_2 = ~~closestPair(P_2)~~ closestPair(P2X, P2Y)

//merge

- let d = min{d_1, d_2} Traverse P1Y: if $x_p > x_L - d$: add p to Strip1
- ~~for all p in P_1: if $x_p > x_L - d$: add p to Strip1~~
- ~~for all p in P_2: if $x_p < x_L + d$: add p to Strip2~~
- ~~sort Strip1, Strip2 by y-coord~~ //Strip1, Strip2 are y-sorted!
- initialize mindist=d
- merge Strip1, Strip2: for next point p,
    - compute its distance to the 5 points that come after it on the other side of the strip
    - if any of these is smaller than mindist, update mindist

- return mindist

Analysis: $T(n) = 2T(n/2) + O(n) \Rightarrow O(n \lg n)$

Hooray!

Almost there..

- We have PX, PY
- We need to:
    - Find the vertical line that splits P in half. How ?
    - Get P1X, P2X. How ?
    - Get P1Y, P2Y. How?

Refining the refined refined merge ?

- Just kidding ..
- Someone must have proven a lower bound of Ω(n lg n) for this problem