Quadtrees
I

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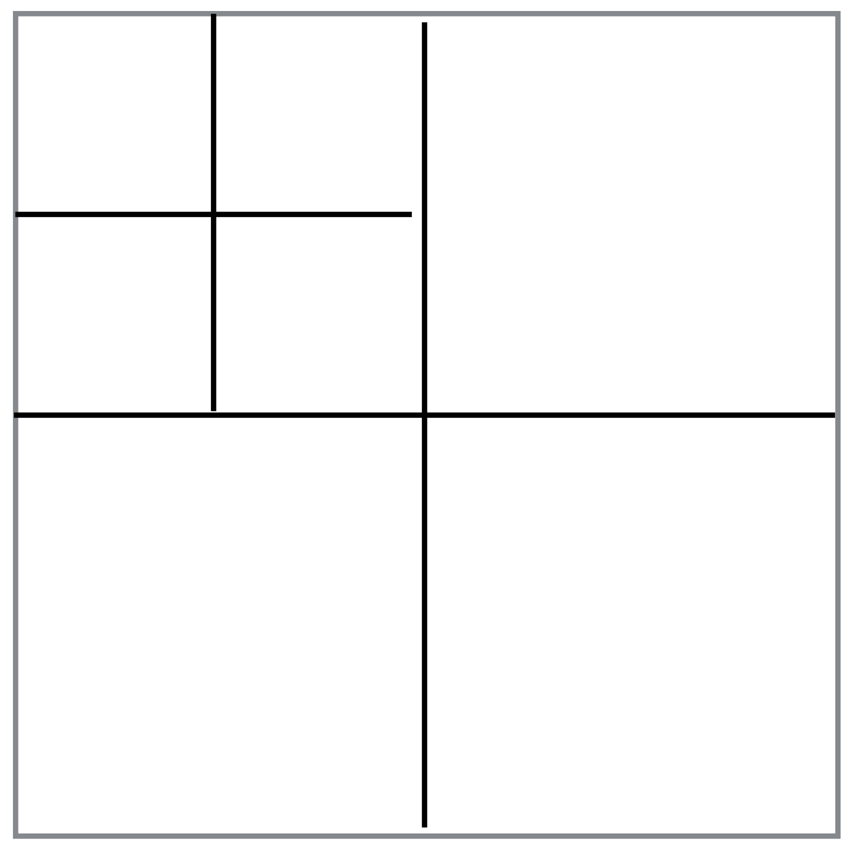
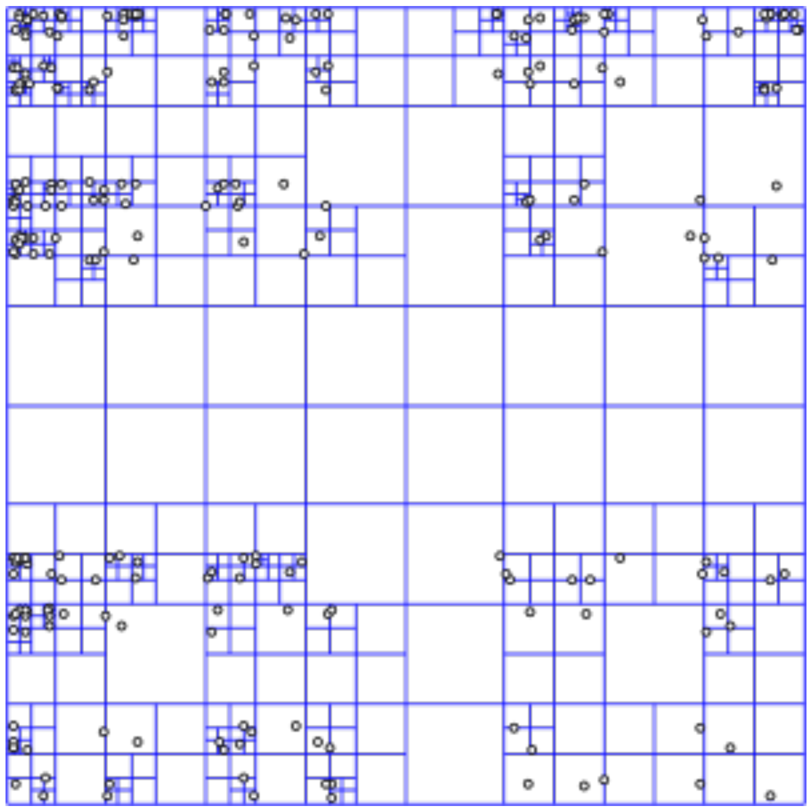
Quadtree

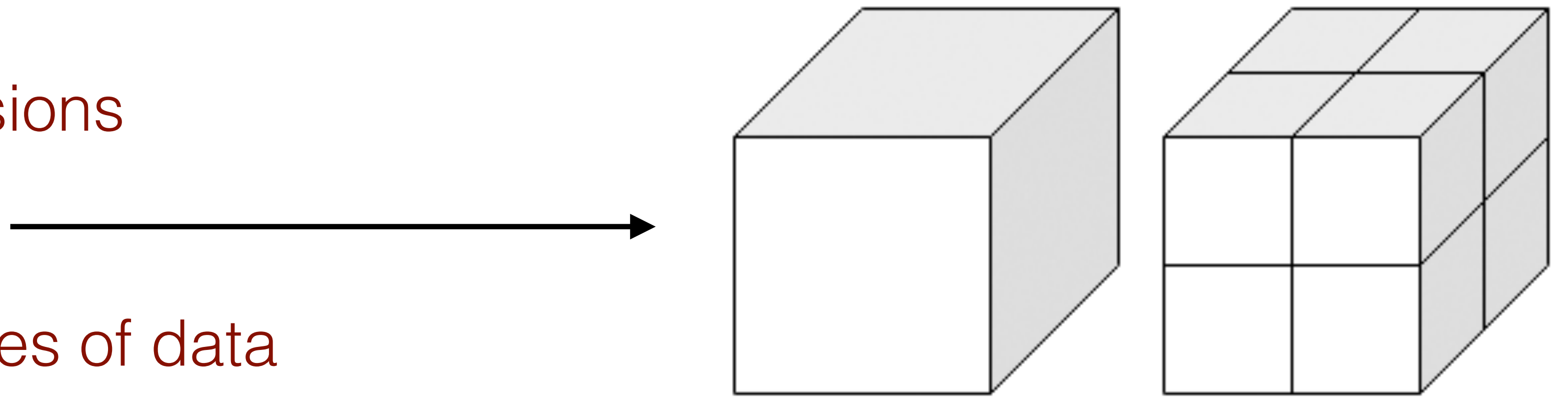
- A data structure that corresponds to a hierarchical subdivision of the plane
- Start with a square (containing inside input data)
    - Divide into 4 equal squares (quadrants)
    - Continue subdividing each quadrant recursively
    - Subdivide a square until it satisfies a stopping condition, usually that a quadrant is “small” enough
        - for e.g. contains at most 1 point

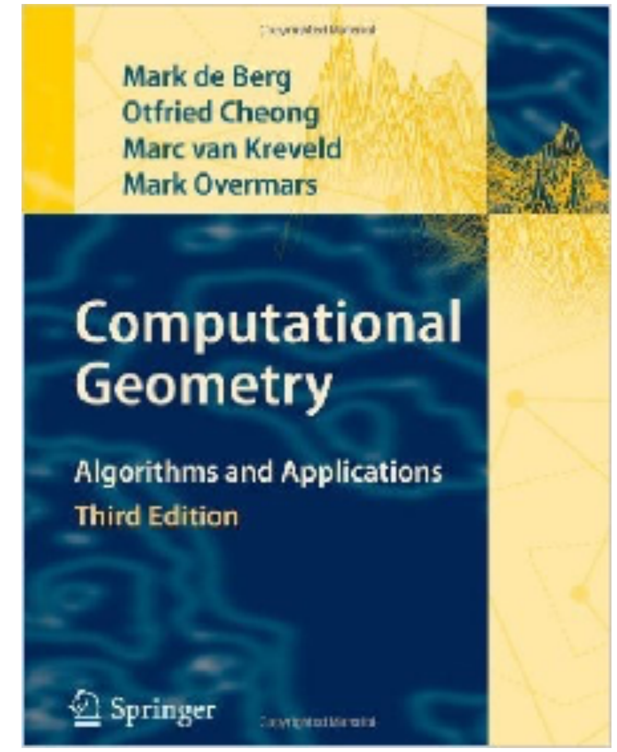




Quadtrees

- Conceptually simple
- Generalizes to >2 dimensions
  - d=3: octree
- Can be built for many types of data
  - points, edges, polygons, images, etc
- Can be used for many different tasks
  - search, point location, neighbors, etc
  - dynamic
- Theoretical bounds not great, but widely used in practice
- LOTS of applications
  - Many variants of quadtrees have been proposed
  - Hundreds of papers





Point-quadtree

Point quadtree

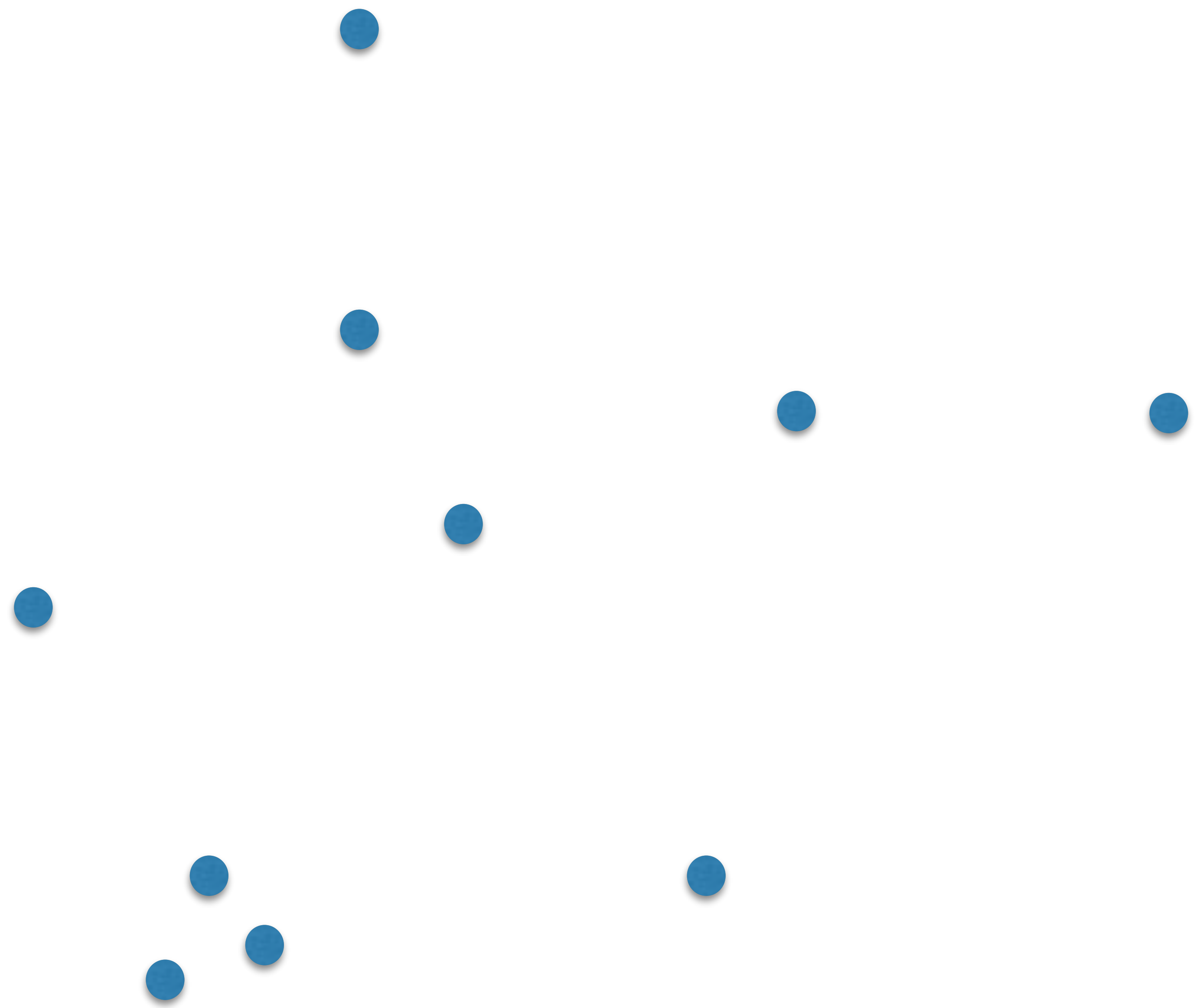
Let P = set of n points in the plane

Problem: Store P in a quadtree such that every square has <= 1 point.

Questions:

1. Size? Height?
2. How to build it and how fast?
3. What can we do with it?

Let P = set of n points in the plane



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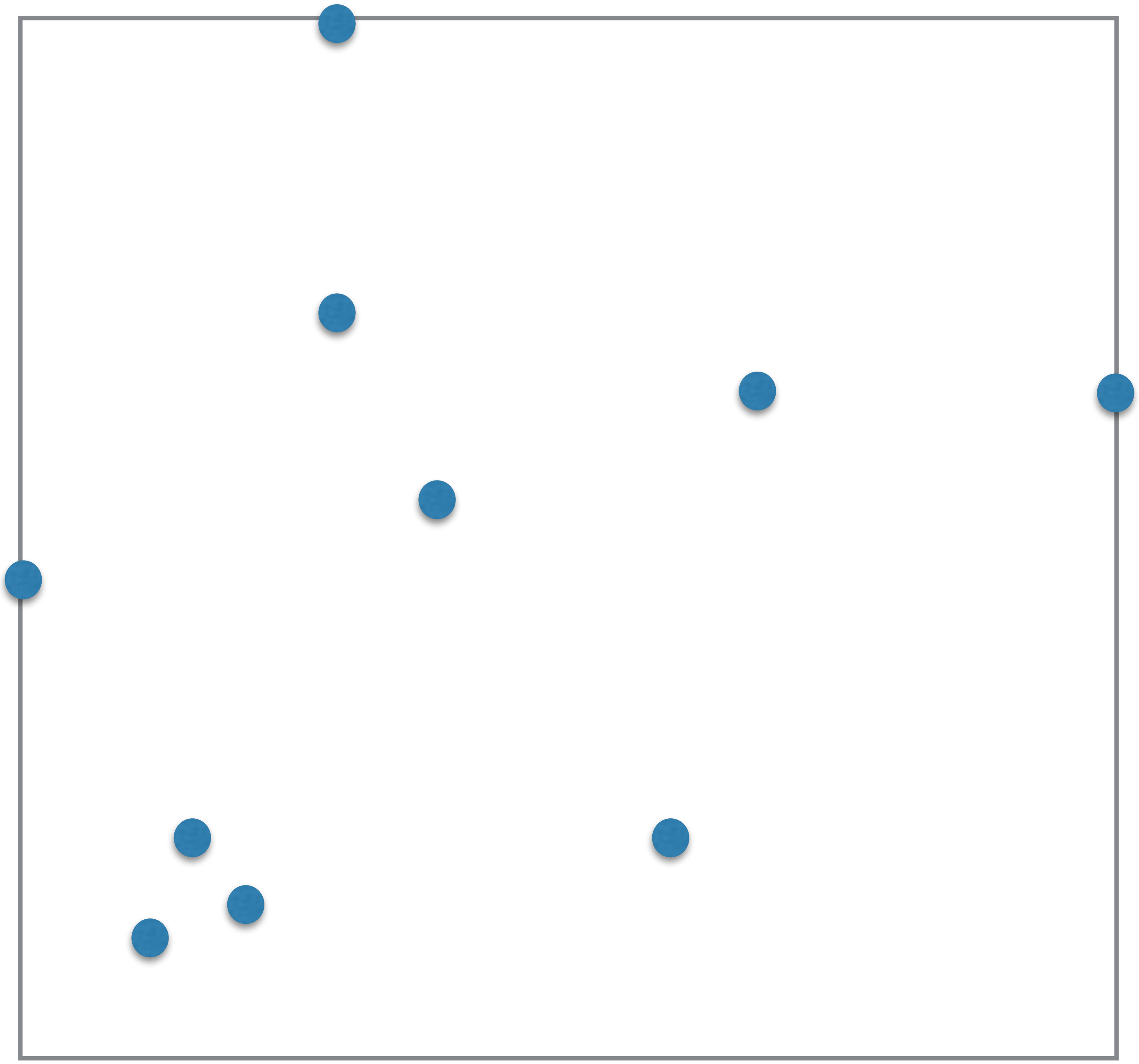

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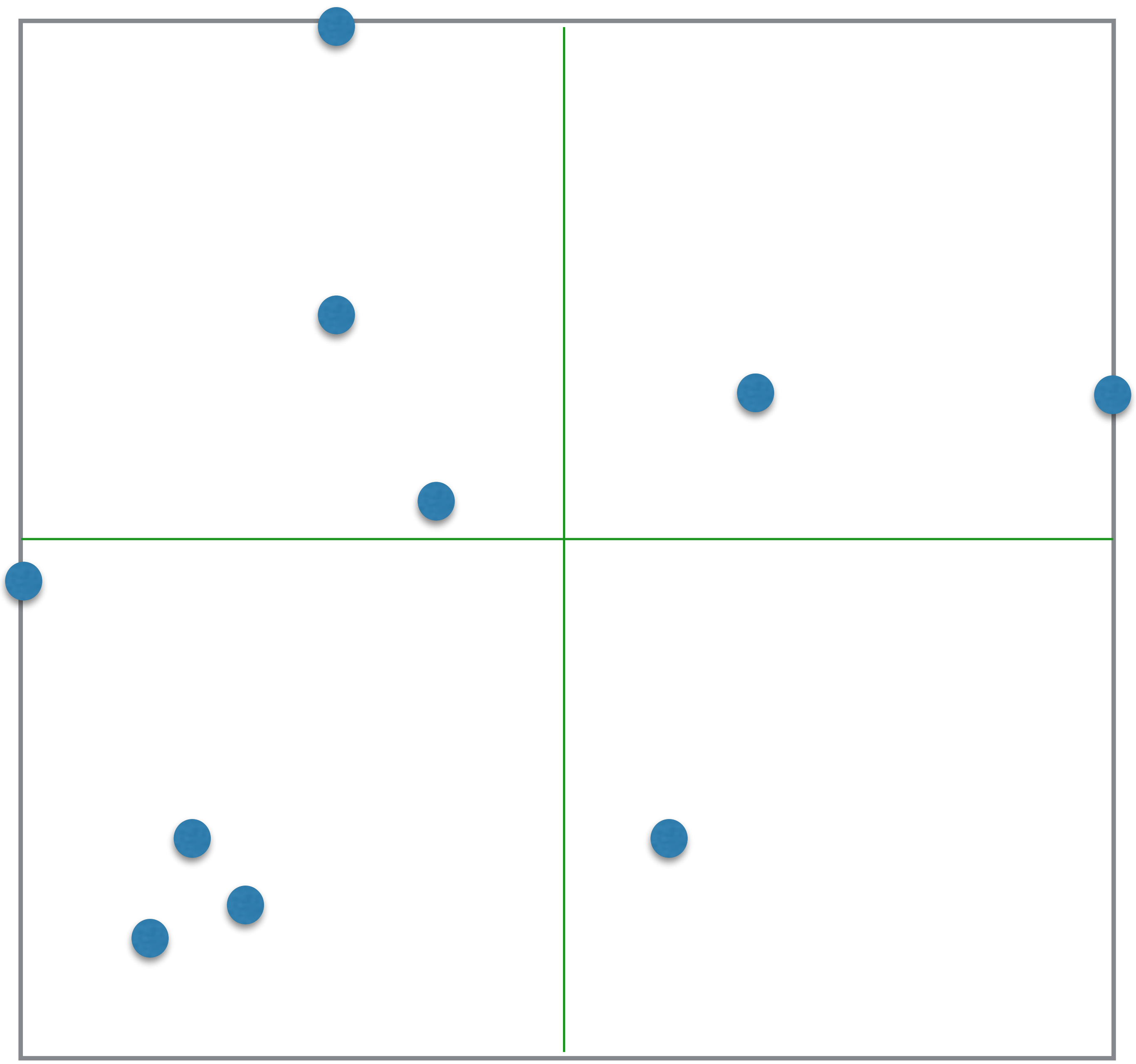

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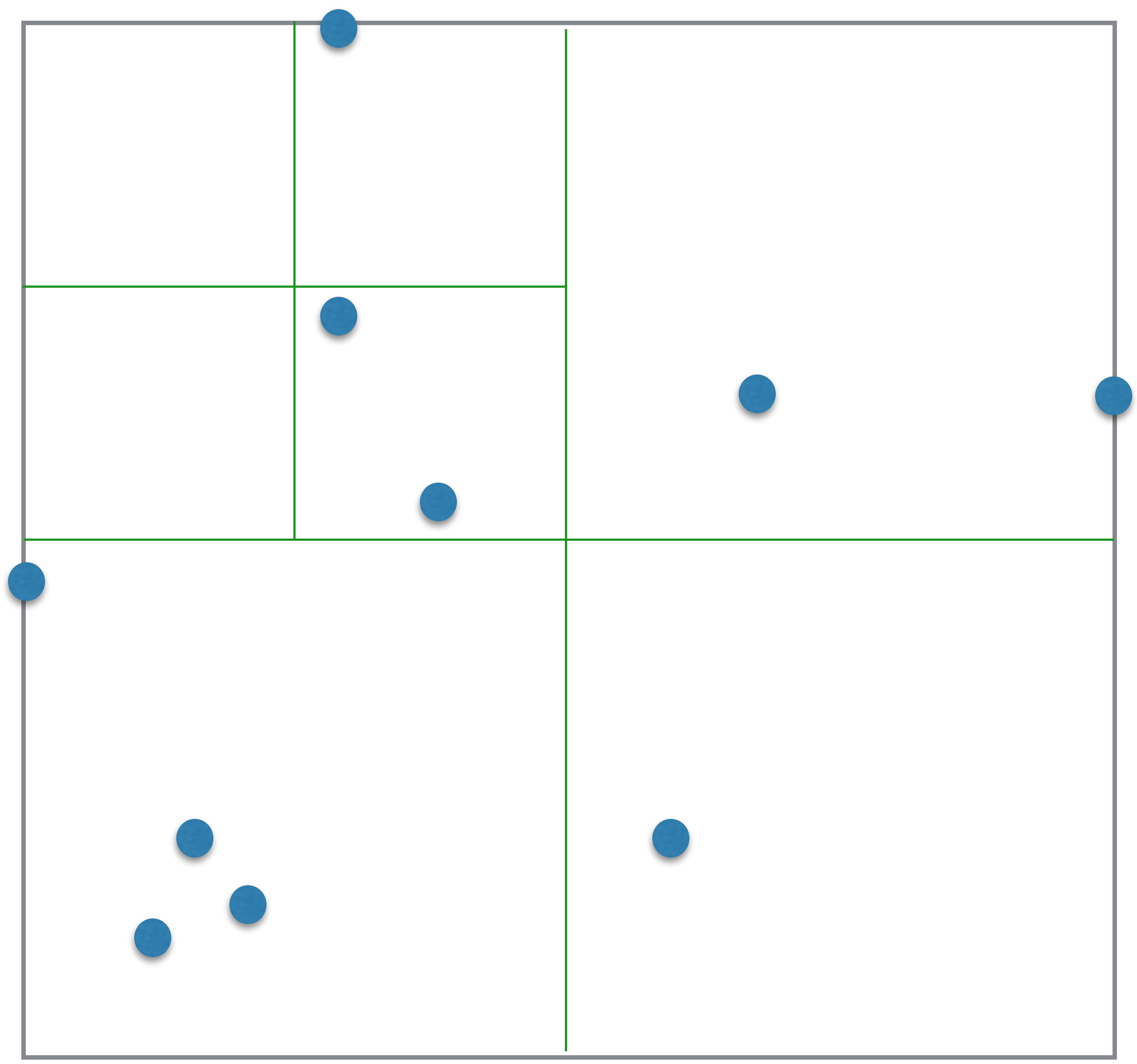

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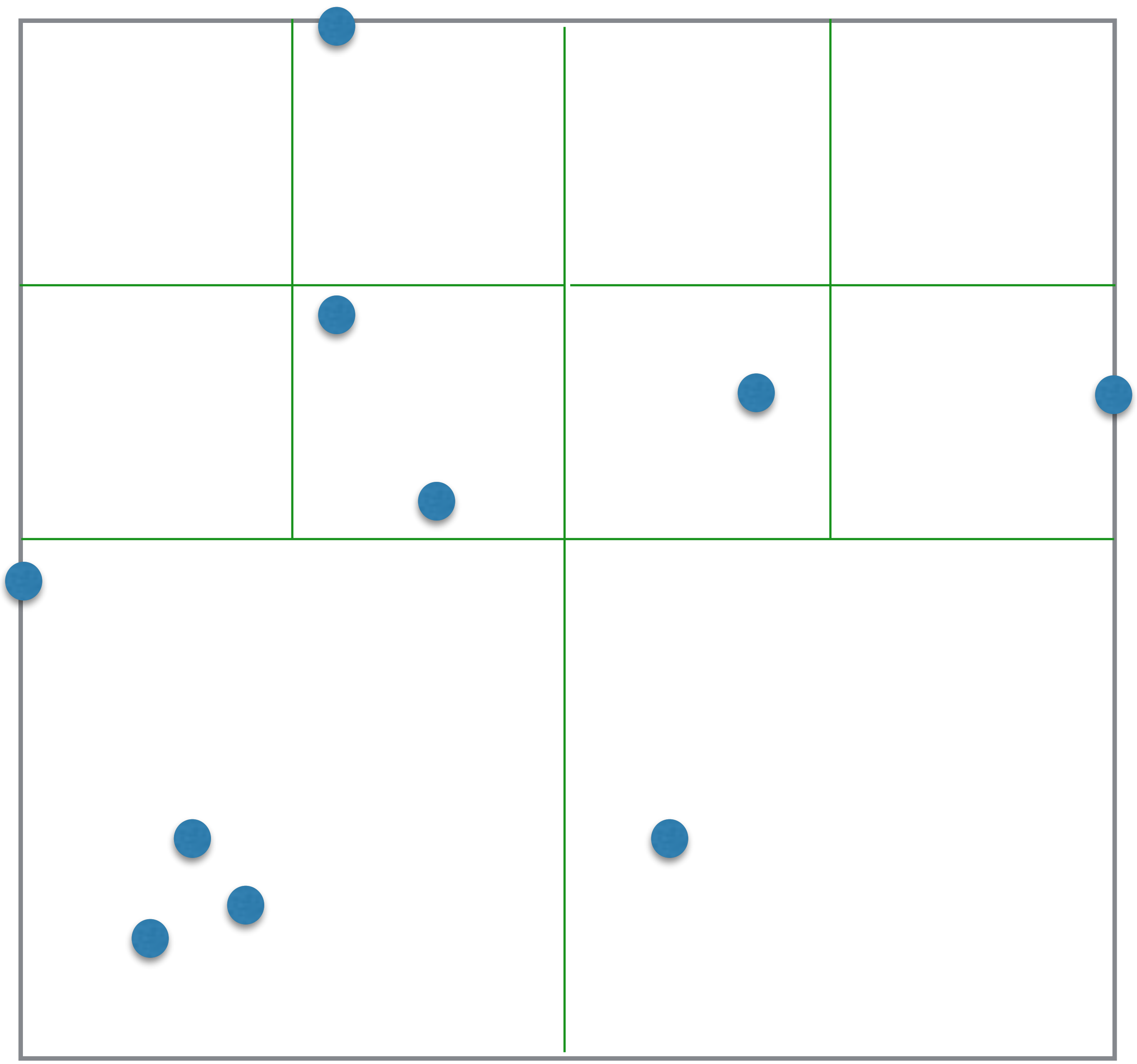

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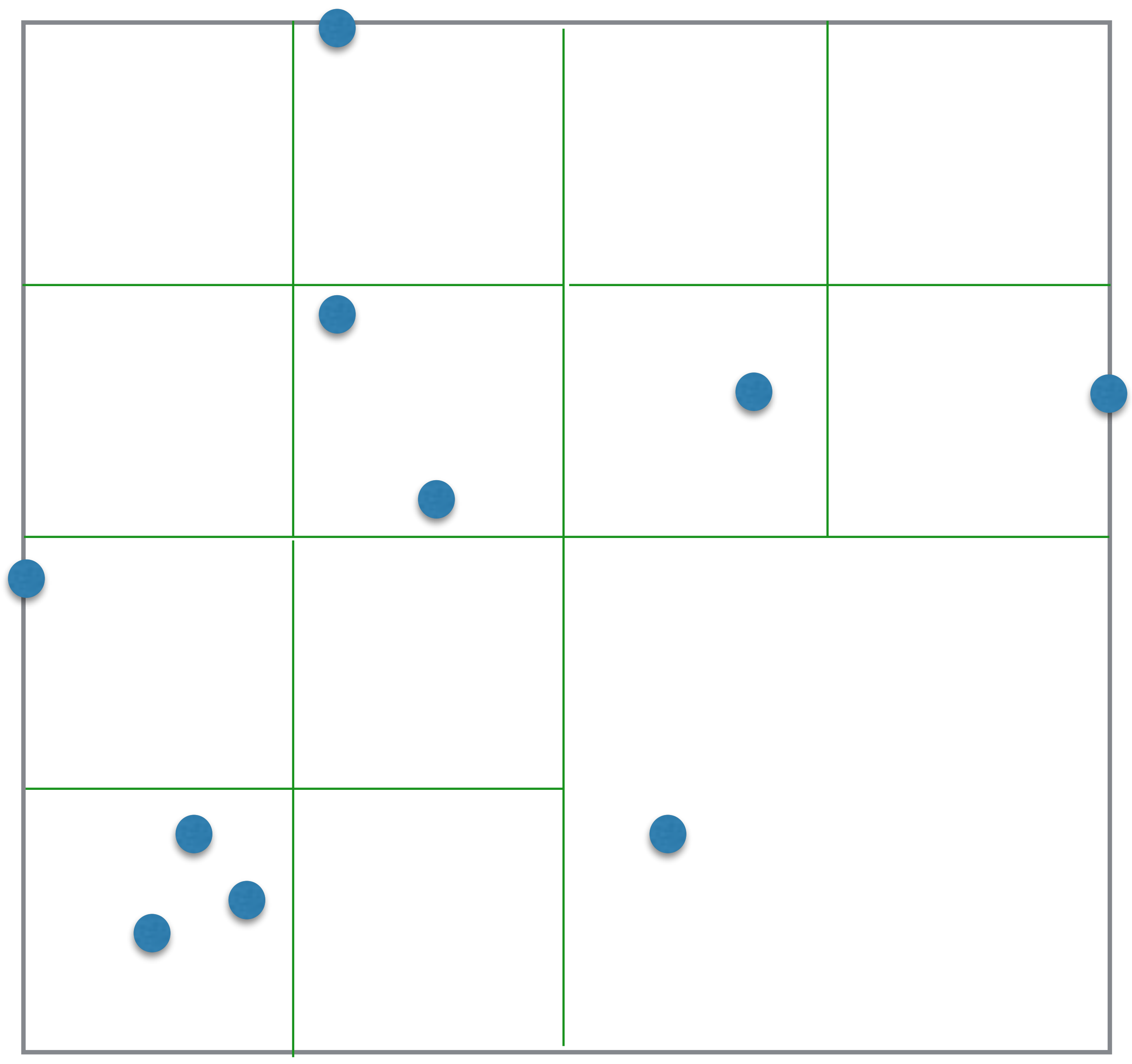

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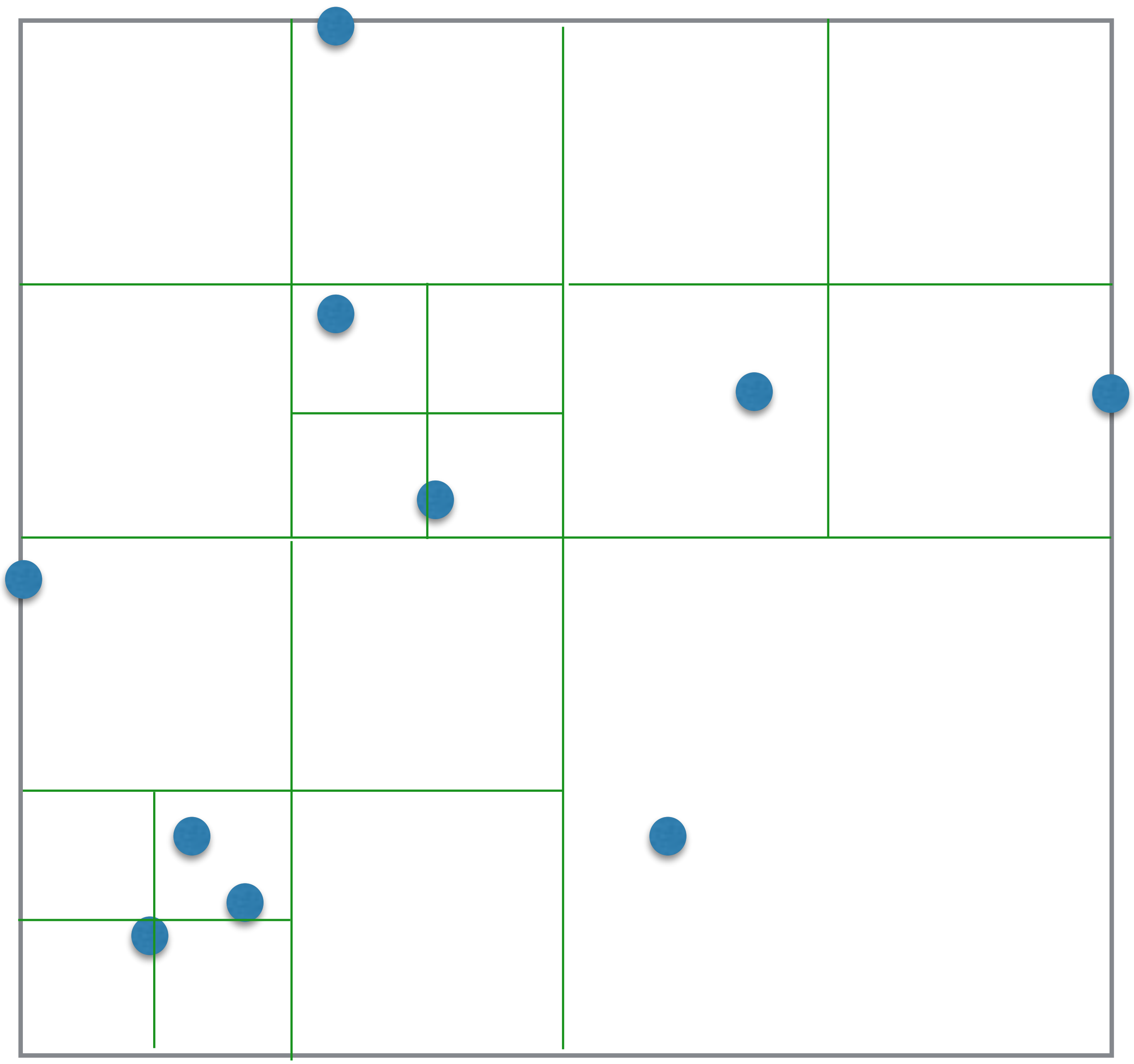

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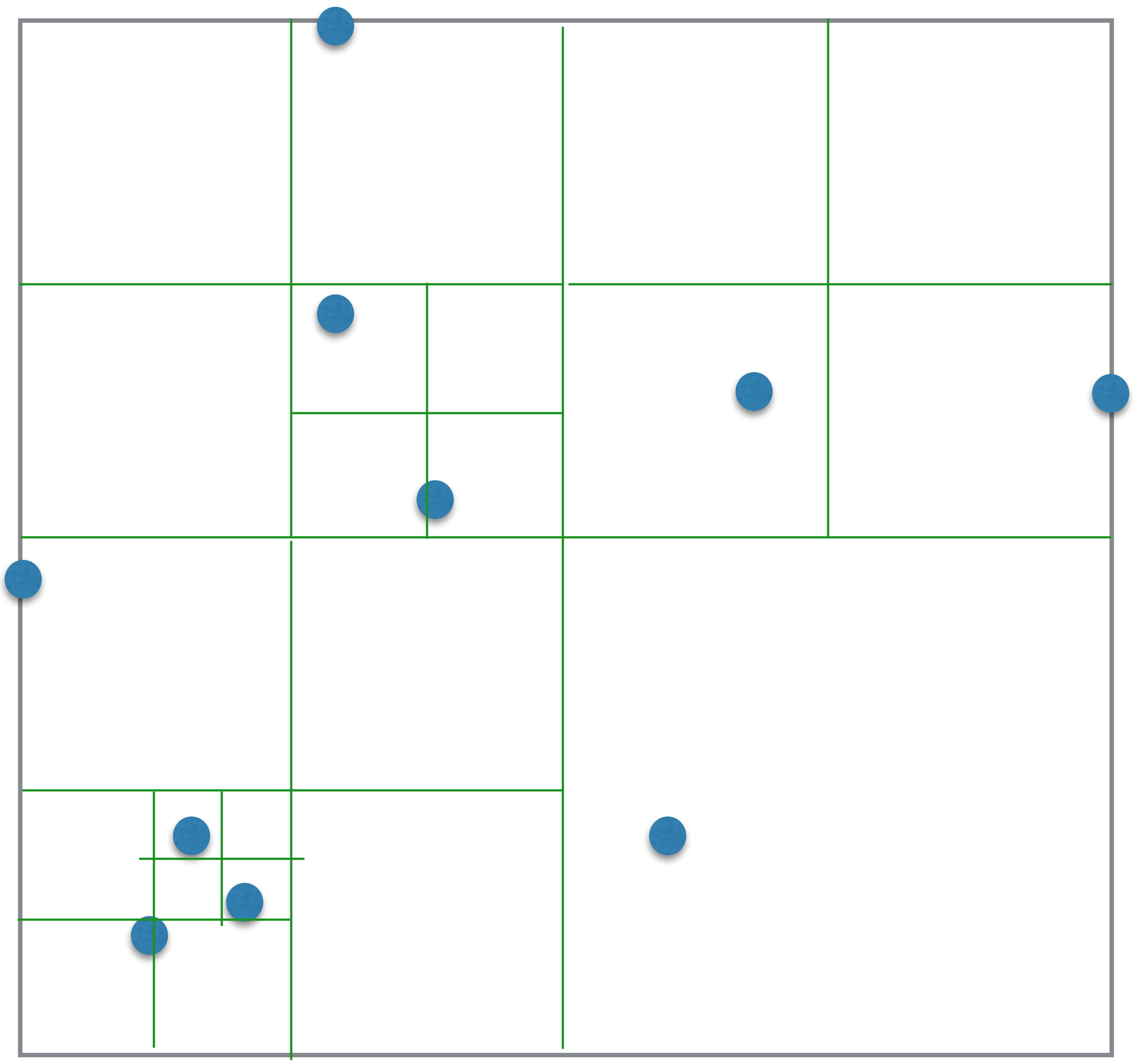

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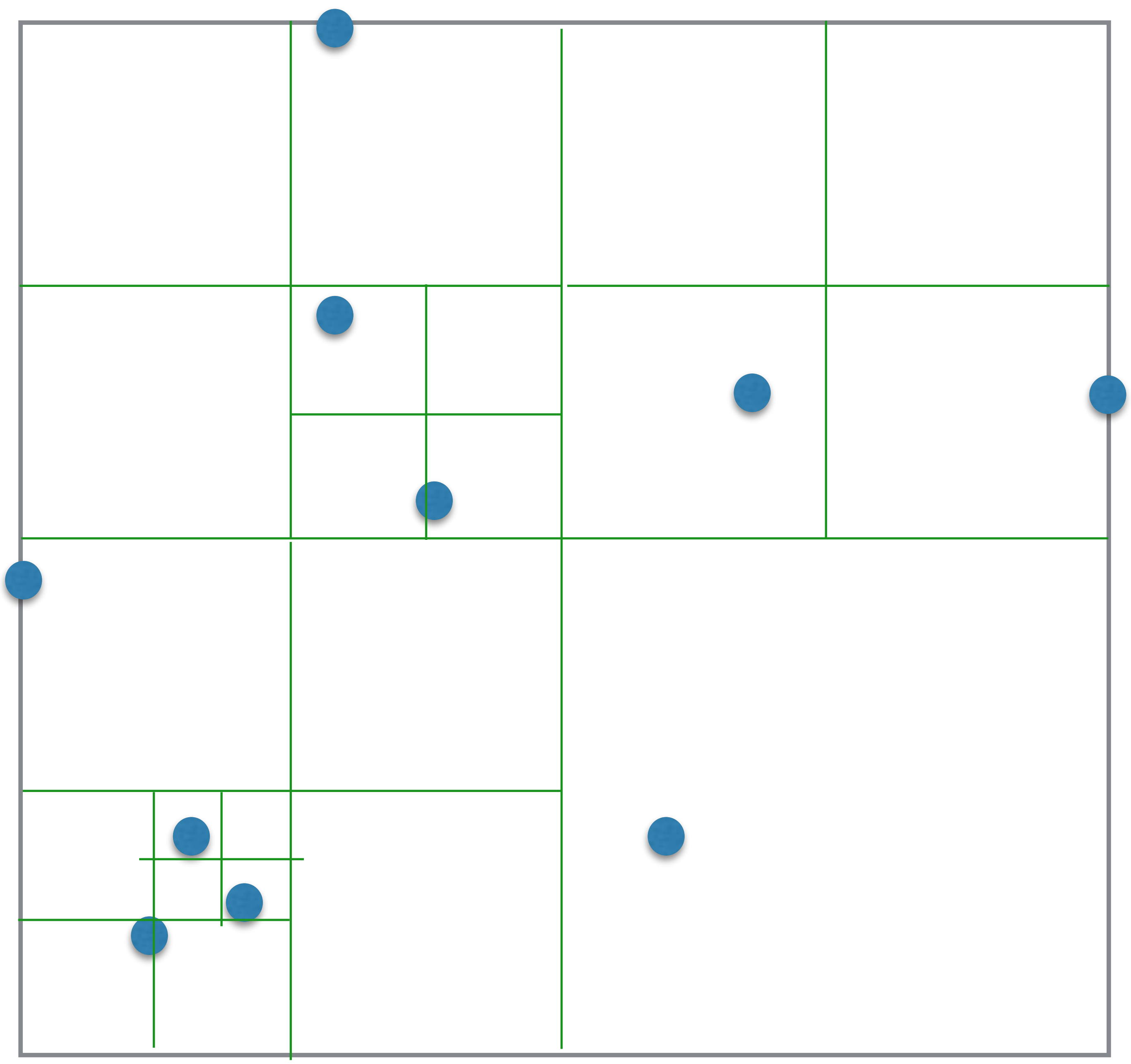

Quadtree: tree corresponding to the subdivision

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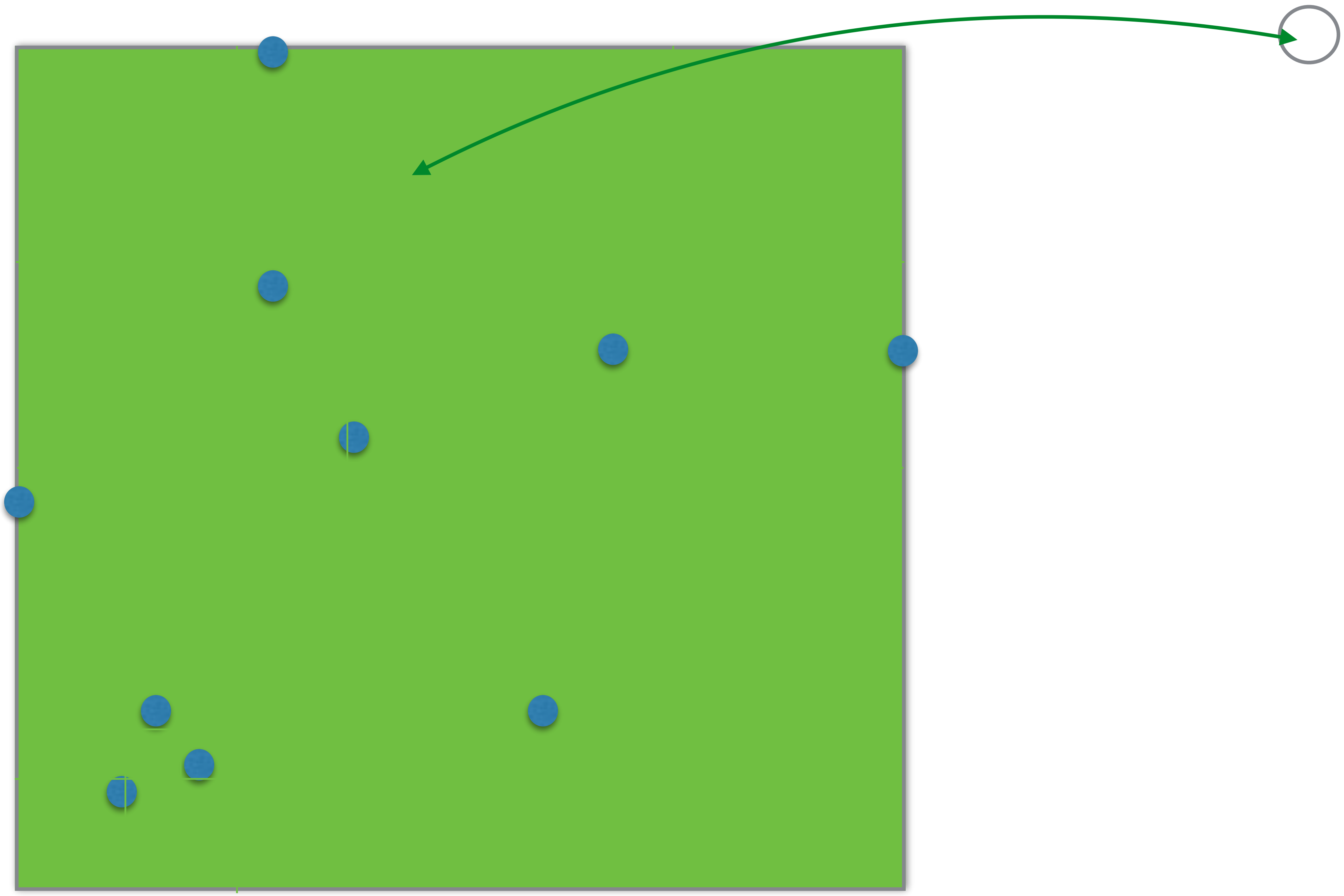

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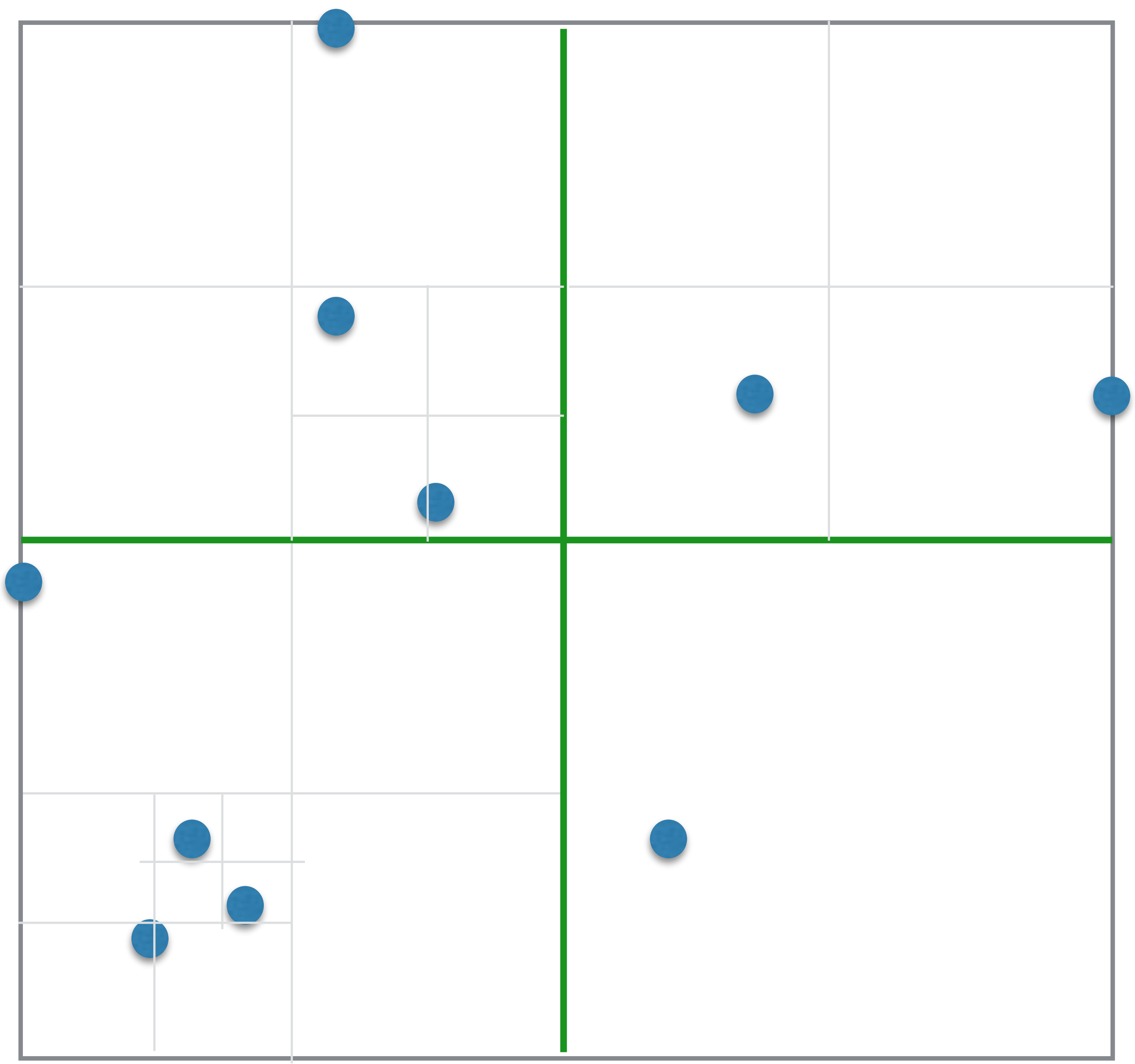

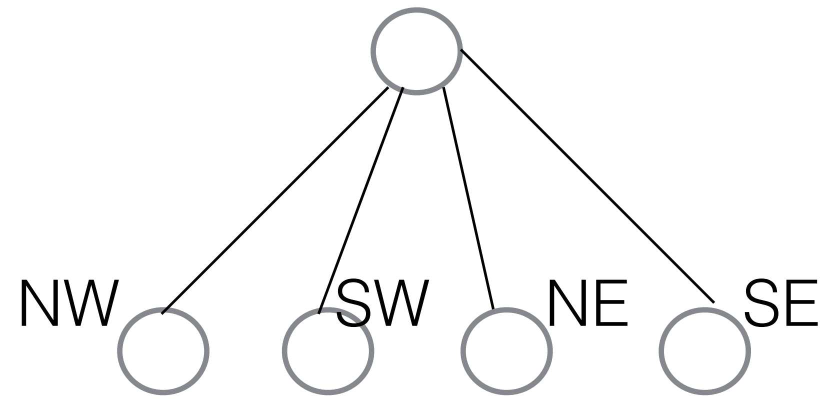

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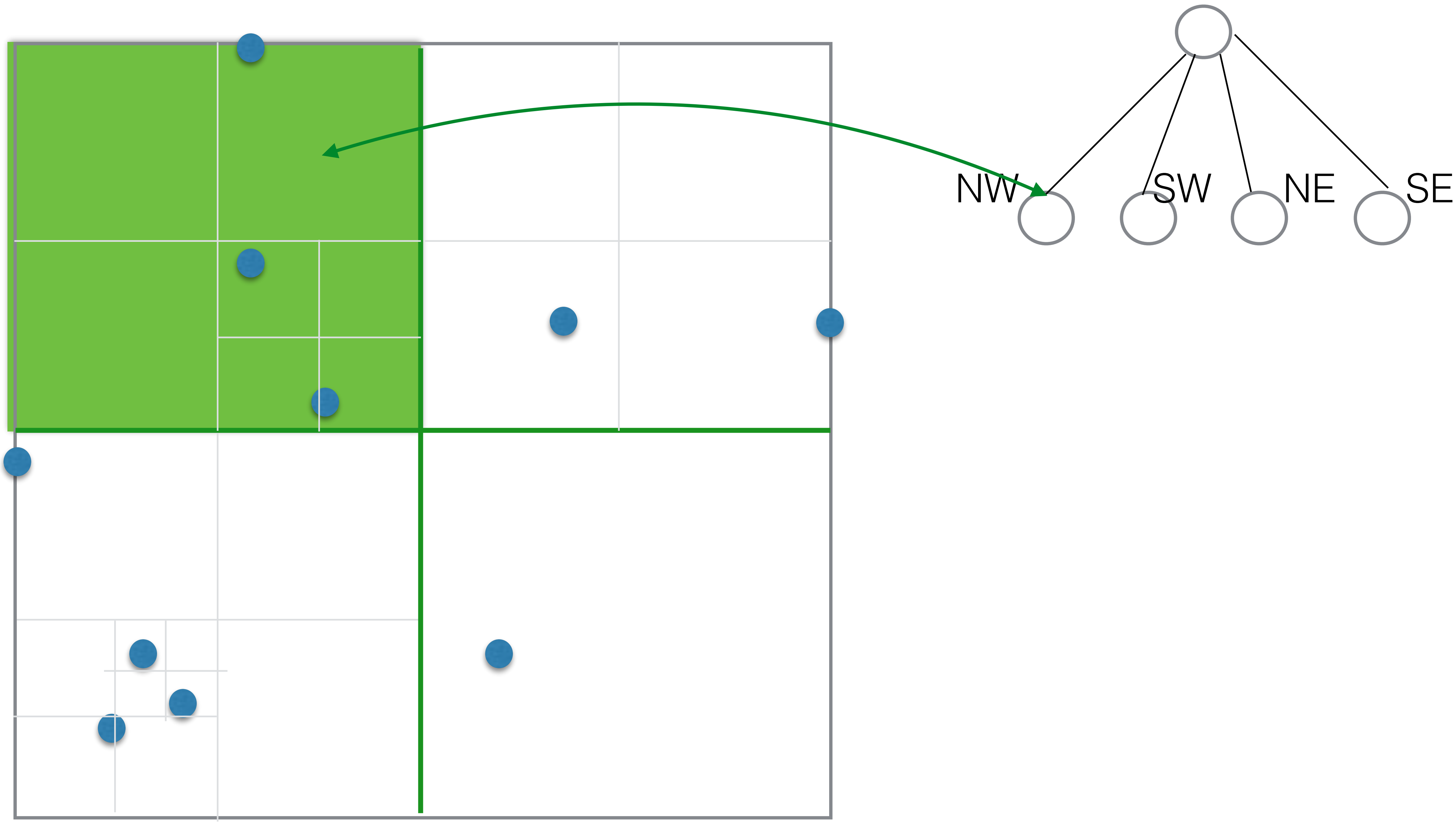

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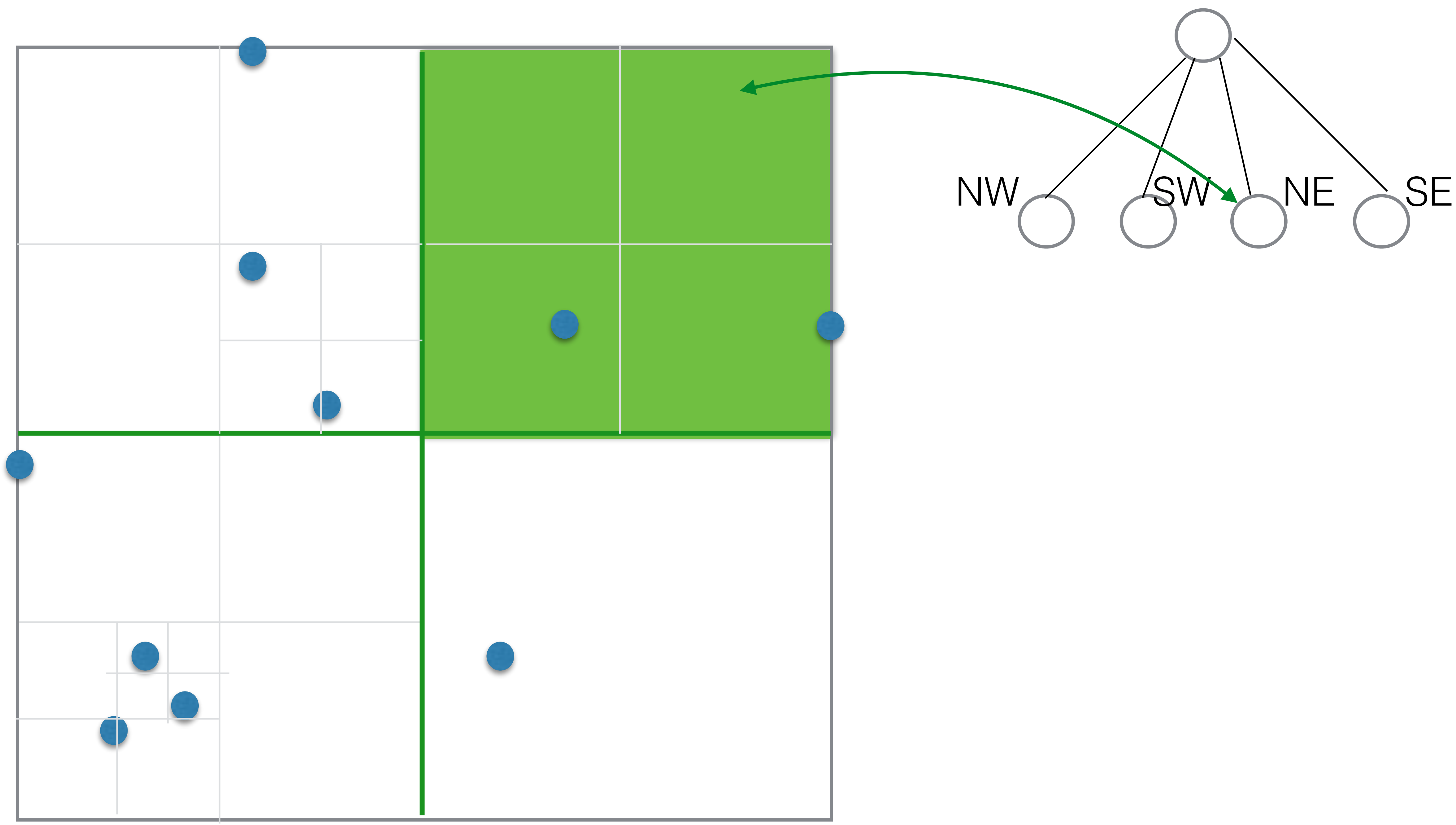

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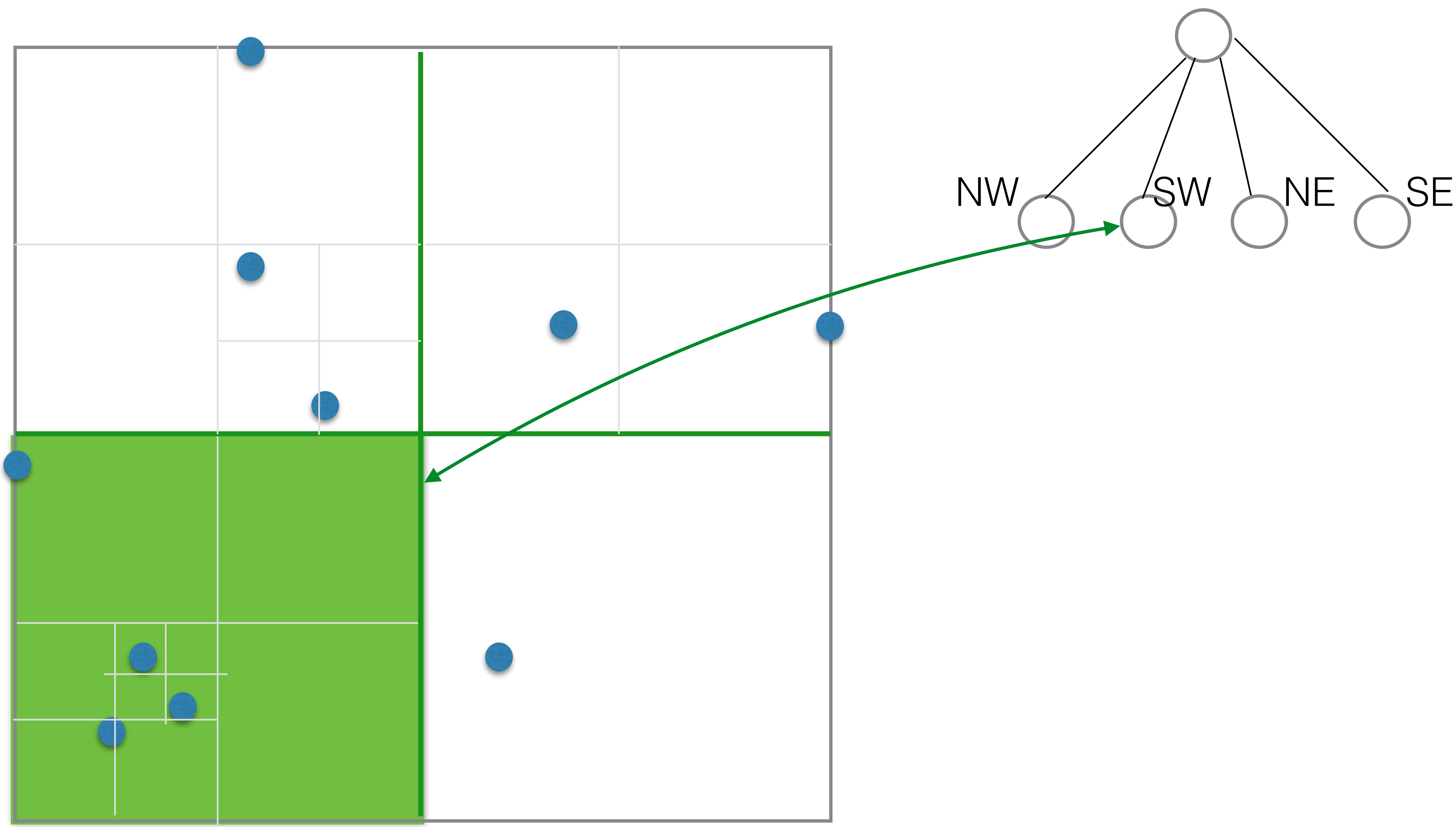

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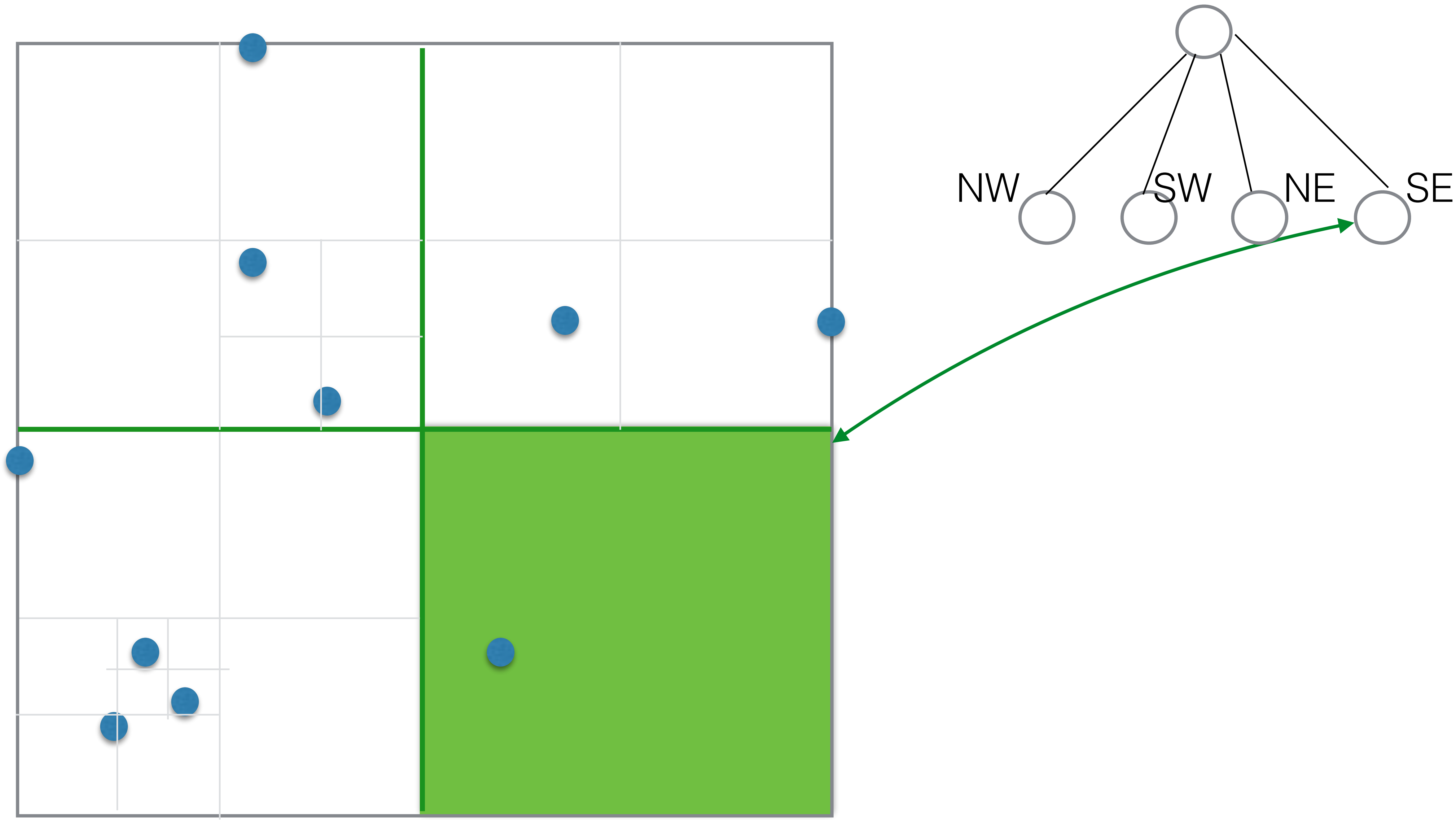

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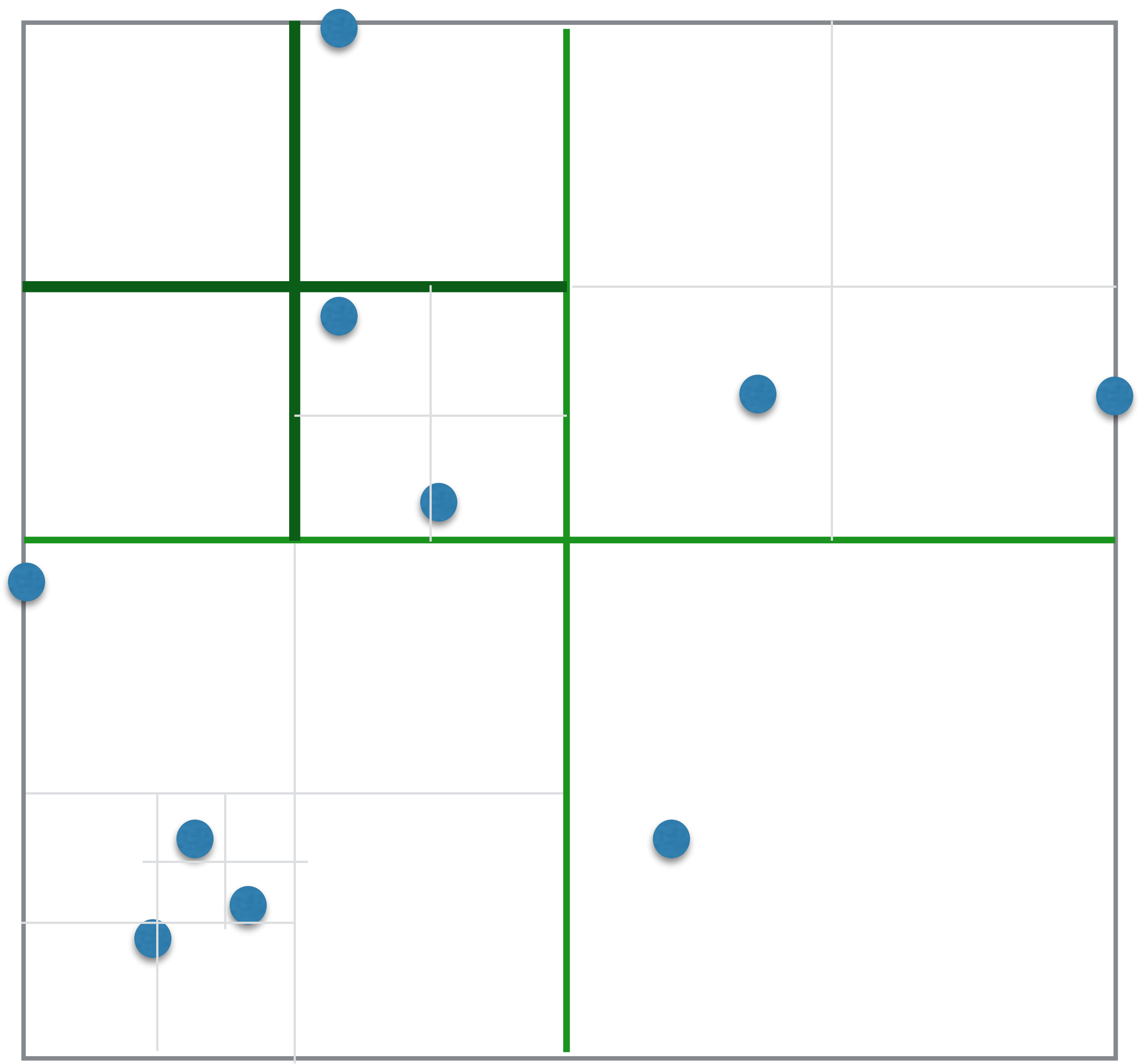

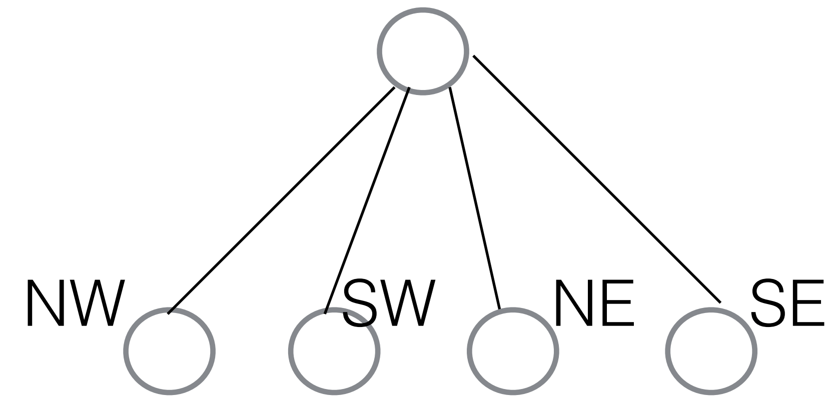

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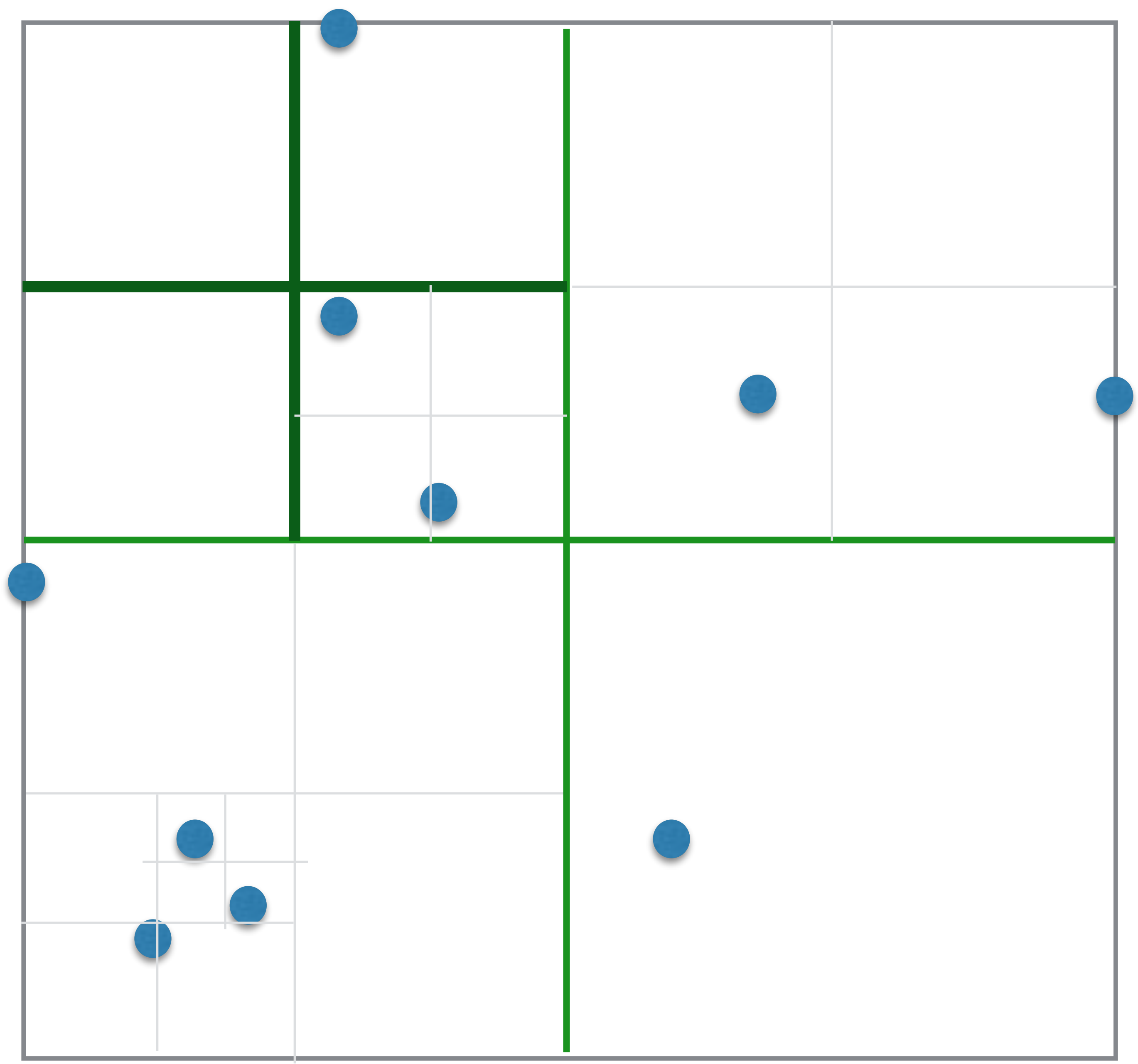

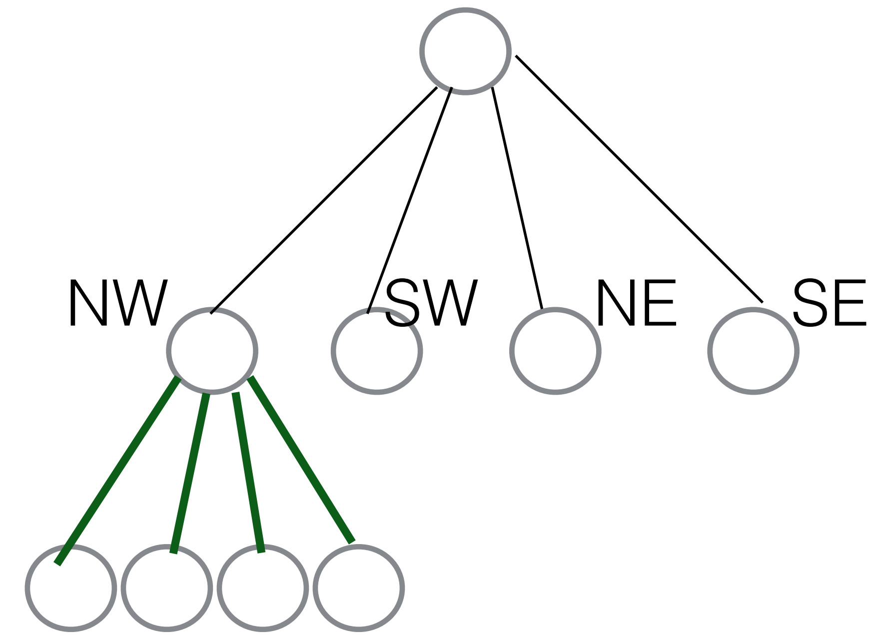

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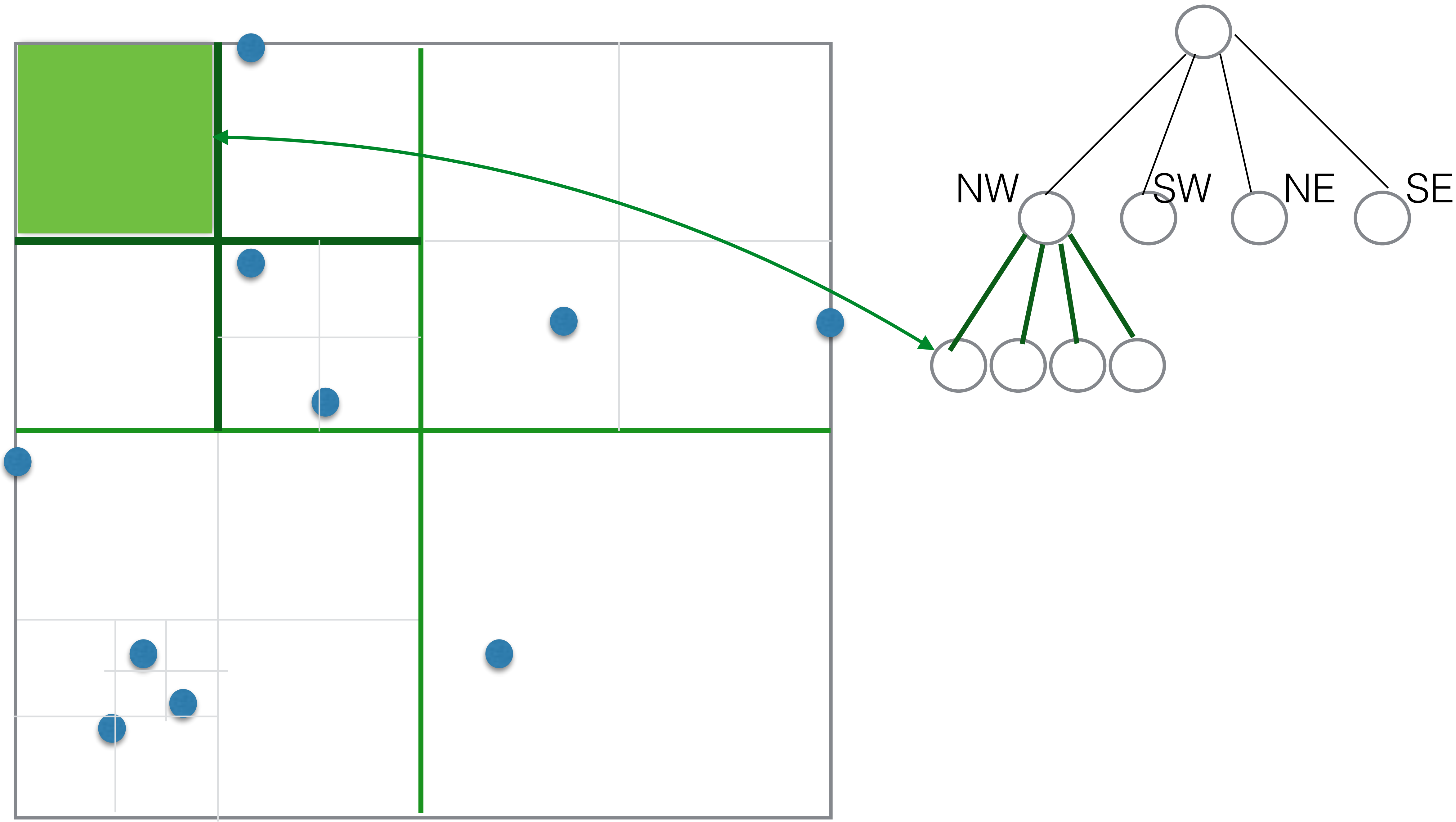

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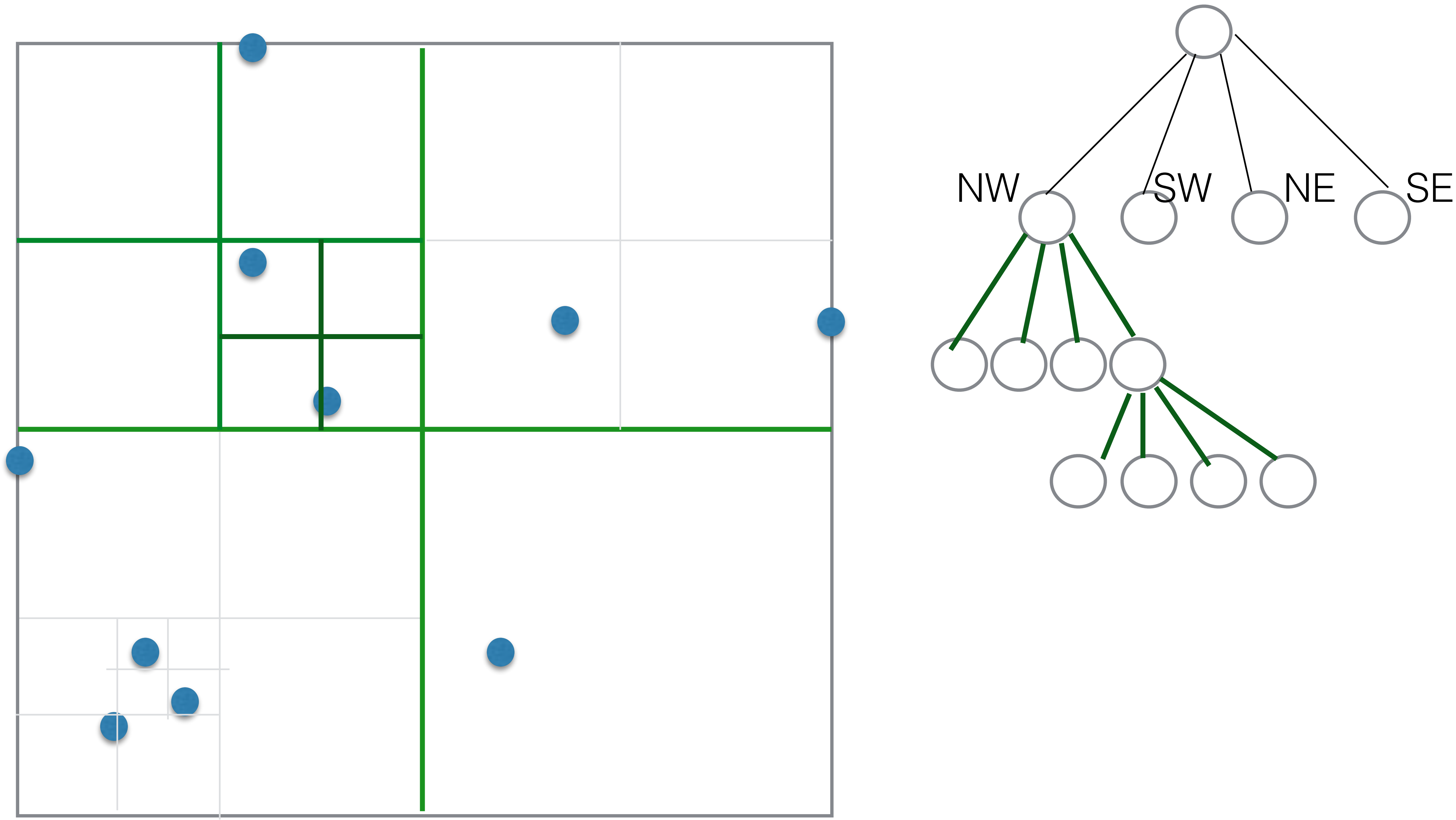

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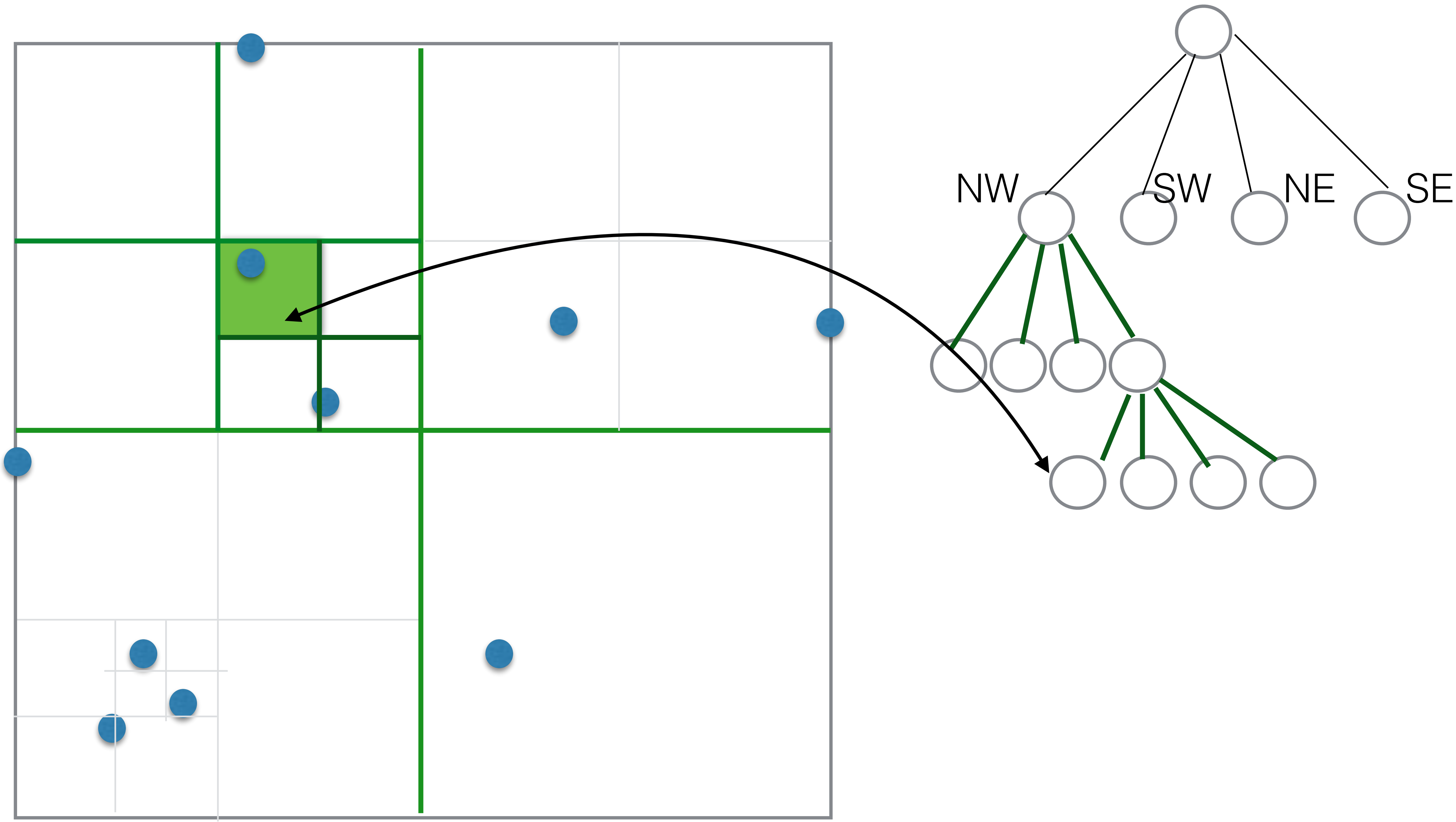

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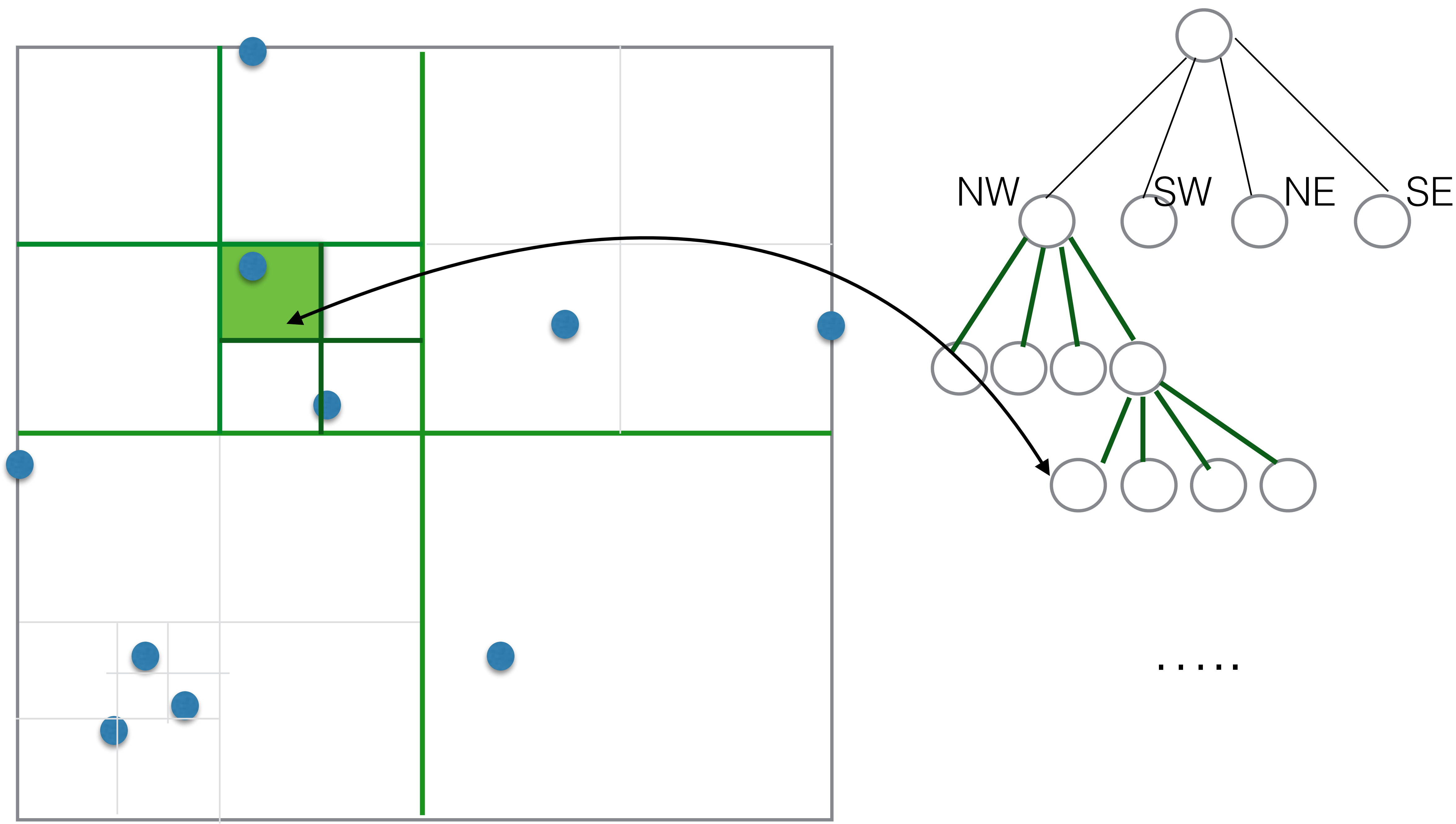

Quadtree: tree corresponding to the subdivision

Exercises

Let P = set of n points in the plane

- Pick n=10 points in the plane and draw their quadtree.
- Show a set of (10) points that have a balanced quadtree.
- Show a set of (10) points that have an unbalanced quadtree.
- Draw the quadtree corresponding to a regular grid
  - how many nodes does it have?
  - how many leaves? height?
- Consider a set of points with a uniform distribution. What can you say about the quadtree ?
- Let's look at sets of 2 points in the plane.
  - Sketch the smallest possible quad tree for two points in the plane.
  - Sketch the largest possible quad tree for two points in the plane.
  - An upper bound for the height of a quadtree for 2 points ????
- What can you say about all points at the same level in the quadtree?

Quadtree size

P = set of n points in the plane

Theorem:

The height of a quadtree storing P is at most lg (s/d) + 3/2 , where s is the side of the original square and d is the distance between the closest pair of points in P.

Proof:

- Each level divides the side of the quadrant into two. After i levels, the side of the quadrant is $s/2^i$
- A quadrant will be split as long as the two closest points will fit inside it.
- In the worst case the closest points will fit diagonally in a quadrant and the "last" split will happen at depth i such that $s\ \mathrm{sqrt}(2)/2^i = d$
- The height of the tree is i+1

- What does this mean?
  - The distance between points can be arbitrarily small, so the height of a quadtree can be arbitrarily large in the worst case

Building a quadtree

Let P = set of n points in the plane

- Let's come up with a (recursive) algorithm to build quadtree of P

  //create quadtree of P and return its root

  buildQuadtree(set of points P, square S)

Building a quadtree

Let P = set of n points in the plane

//create quadtree of P and return its root

buildQuadtree(set of points P, square S)

- if P has at most one point:
    - build a leaf node , store P in it, and return node
- else
    - partition S into 4 quadrants S1, S2, S3, S4 and use them to partition P into P1, P2, P3, P4
    - create a node
    - node ->child1 = buildQuadtree(P1, S1)
    - node ->child2 = buildQuadtree(P2, S2)
    - node ->child3 = buildQuadtree(P3, S3)
    - node ->child4 = buildQuadtree(P4, S4)
    - return node

Building a quadtree

Let P = set of n points in the plane

```
//create quadtree of P and return its root
buildQuadtree(set of points P, square S)
  • if P has at most one point:
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      • node ->child3 = buildQuadtree(P3, S3)
      • node ->child4 = buildQuadtree(P4, S4)
      • return node
```

How long does this take, function of n and height h?

Analysis

- The logic
  - Total time = total time to partition + total time in recursion
- We'll show that
  - Partition: $O(n \times h)$
  - Recursion: $O(n \times h)$

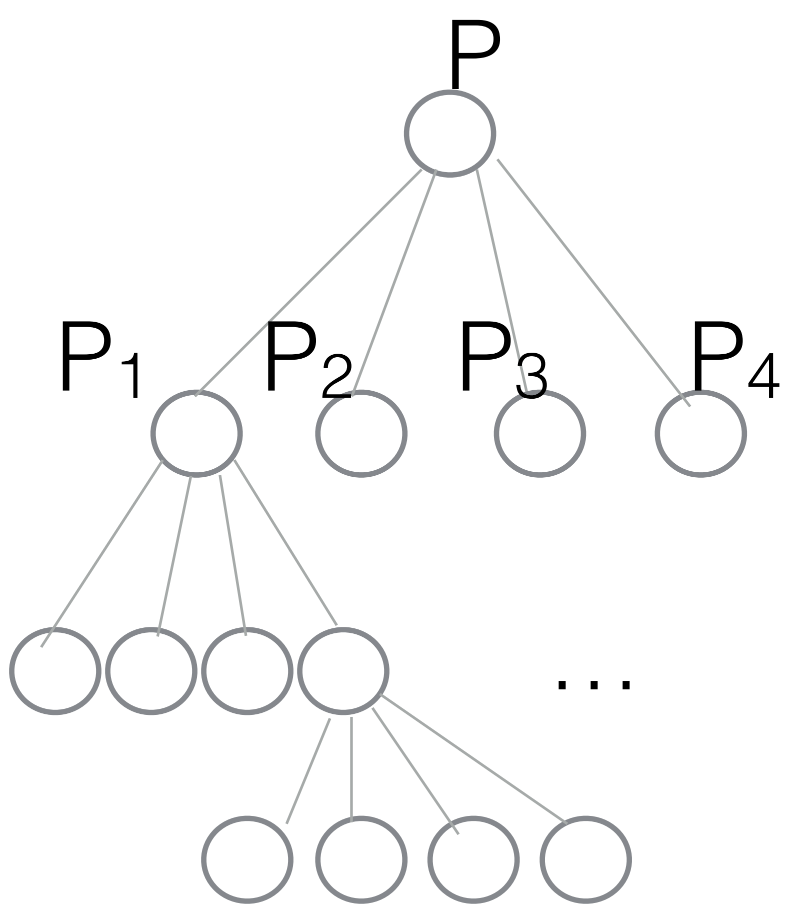
Theorem:

A quadtree for a set P of points in the plane can be built in $O(n \times h)$ time.

Partitioning

Let P = set of n points in the plane

A quadtree for P of height h



Partition P into P_1, P_2, P_3 P_4 takes $O(|P|) = O(n)$

$P_1 + P_2 + P_3 + P_4 = P$

Partition P_1, P_2, P_3 P_4 takes $O(|P|) = O(n)$

...

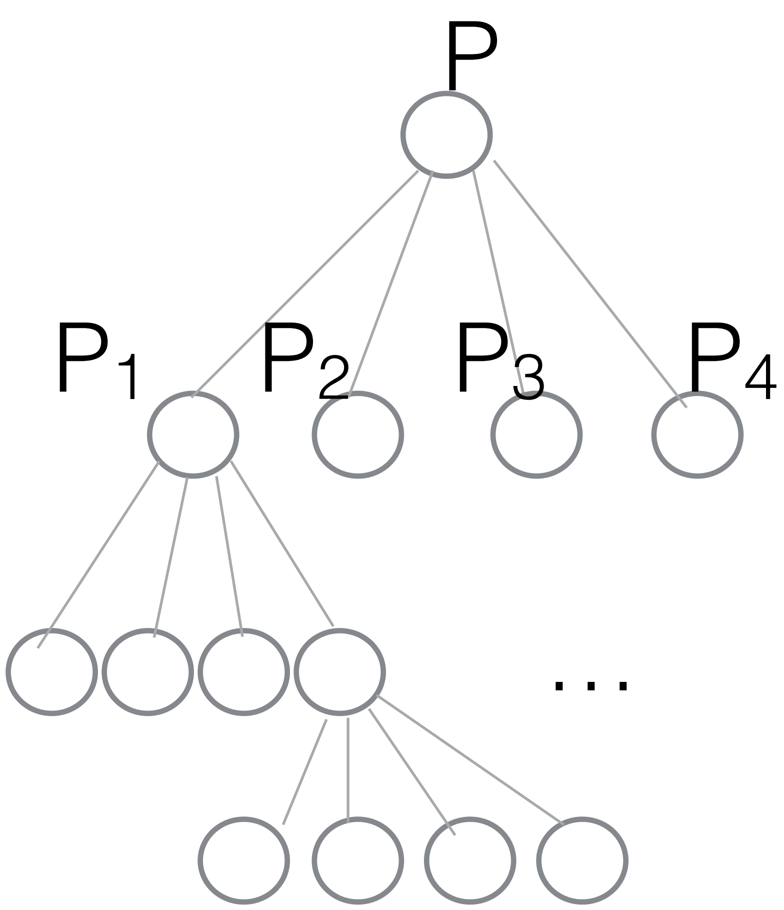
The time to partition, at every level, is O(n)

$O(h \times n)$ total

Recursion

Let P = set of n points in the plane

A quadtree for P of height h

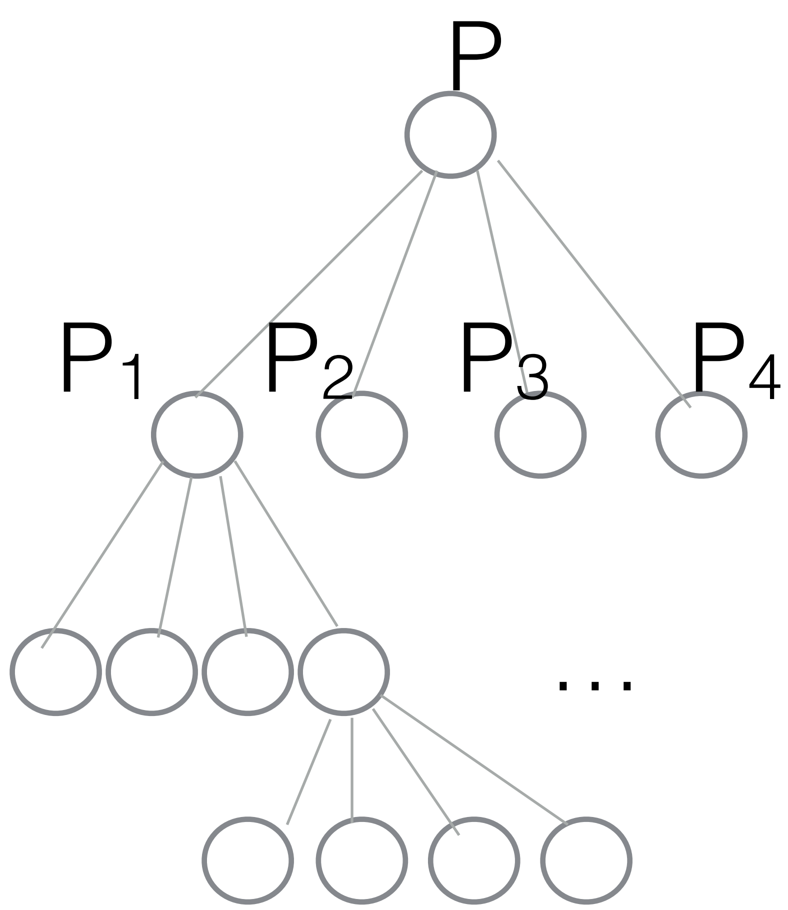


- Every recursive call creates a node
- How many nodes?
    - The number of nodes can be unbounded.
    - We want to express nb.nodes as function of height h.

Recursion

Let P = set of n points in the plane

A quadtree for P of height h



- Every recursive call creates a node
- How many nodes?
  - nodes = internal nodes + leaves

    N = I + L
  - We can find a relation between I and L
    - Each internal node has 4 children.
    - It can be shown that L = 3 I + 1

      (proof by induction)

Recursion

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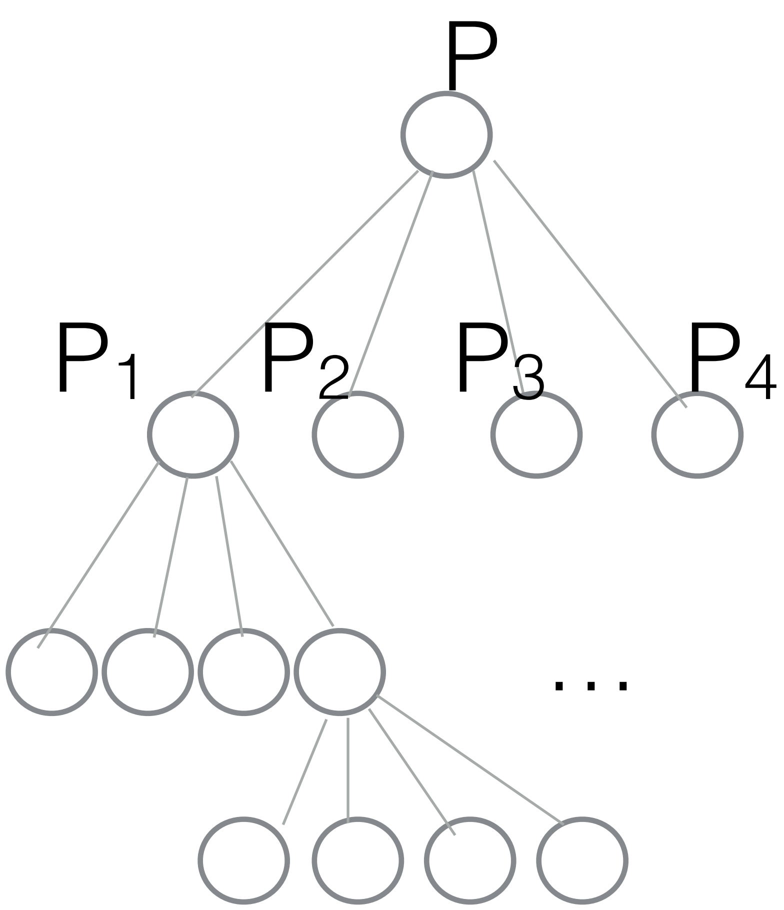
A quadtree for P of height h



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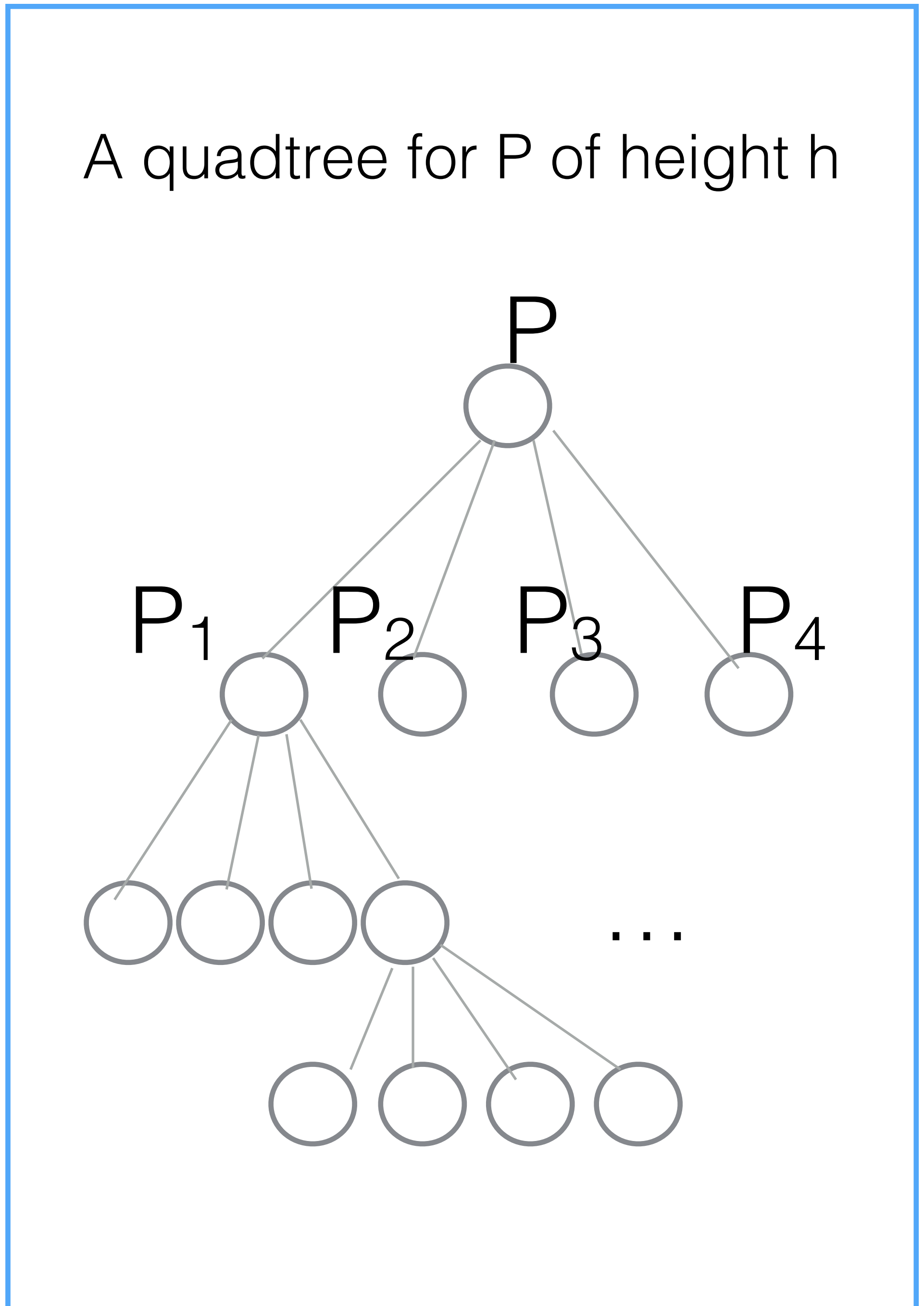
      (proof by induction)
  - It follows that N = I + L = 4I + 1

Building a quadtree



- How many internal nodes?
  - Can be unbounded
  - Want to express function of h
  - The usual argument does not work
    - each leaf contains at most one point
    - best case: no empty leaves
    - worst case: many empty leaves, many internal nodes
  - At each level, each internal node contains at least 2 points

    => O(n) internal nodes per level

O(n x h) nodes

Summary

Theorem:

A quadtree for a set P of points in the plane:

- has height $h = O(\lg (1/d))$ (where d is closest distance)
- has $O(h \times n)$ nodes; and
- can be built in $O(h \times n)$ time.

- Theoretical worst case:
  - height and size are unbounded
- In practice:
  - often $h = O(n)$ ==> size = $O(n^2)$, build time is $O(n^2)$
  - For sets of points that are uniformly distributed, quadtrees have height $h = O(\lg n)$, size $O(n \lg n)$ and can be built in $O(n \lg n)$ time.

Compressed (point) quadtrees

Exercise

Let P = set of n points in the plane

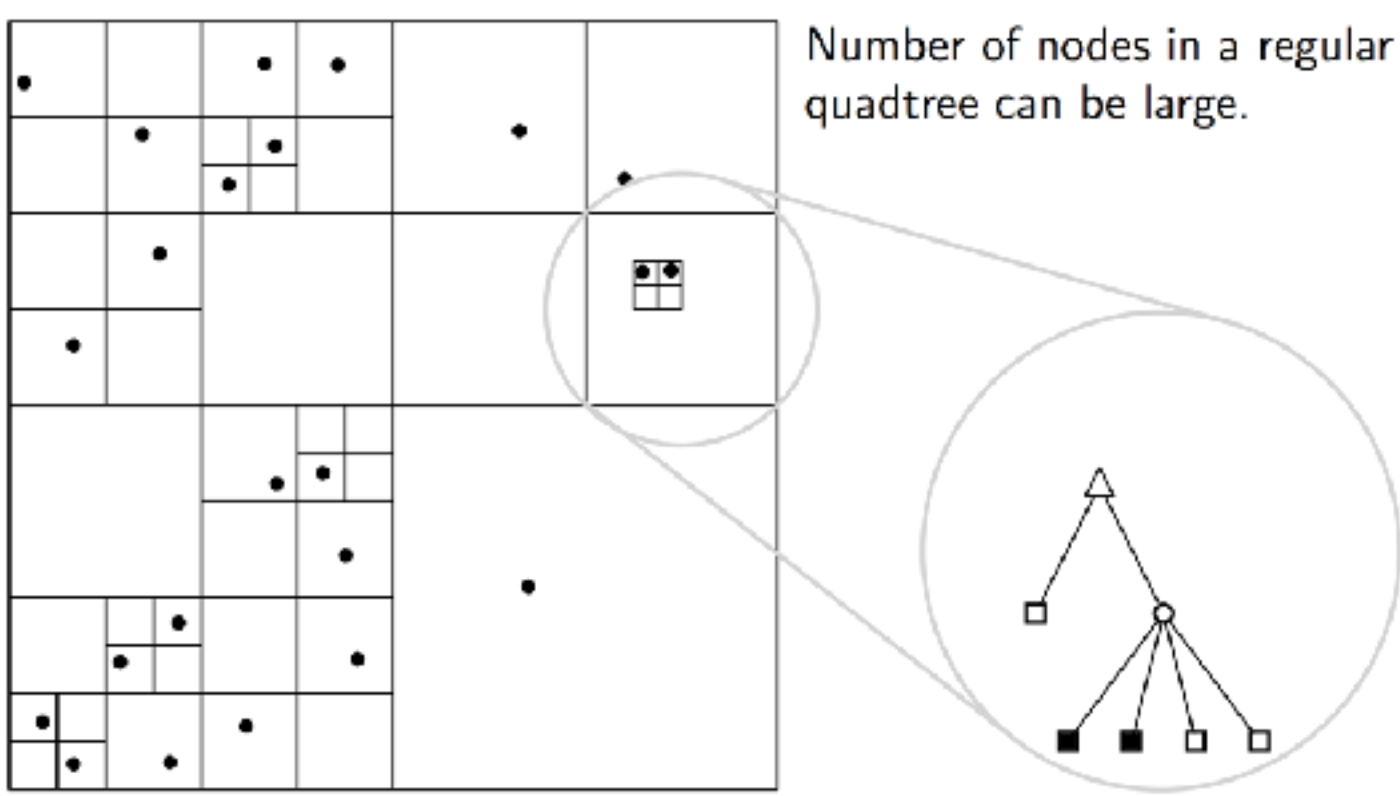
- Draw a quadtree of arbitrarily large size corresponding to a small set of points in the plane (pick n=2 or n=3).
  - How many leaves are empty / non-empty?
  - Why is the size of the quadtree super-linear?

- Compress the quadtree as follows:
  - Compress paths of nodes with 3 empty children into one node
  - This node is called a *donut*
  - *A* node may have 5 children, an empty *donut* + 4 regular quadrants

Compressed quadtrees

Let P = set of n points in the plane

- A compressed quadtree is a regular quadtree where paths of nodes with 3 empty children are compressed into one node (called: donut)
- A node may have 5 children, an empty *donut* + 4 regular quadrants



Compressed quadtrees

- A compressed quadtree is a regular quadtree where paths of nodes with 3 empty children are compressed into one node (called: donut)
- A node may have 5 children, an empty *donut* + 4 regular quadrants

- What does this mean in terms of size?

Theorem: A compressed quadtree has O(n) nodes and h=O(n) height.

- Proof idea: For each leaf that's empty and for each donut, there exists one sibling leaf that's not empty. The number of non-empty leaves is n.

Applications of quadtrees

- Hundreds of papers
- Specialized quadtrees
  - customized for specific types of data (images, edges, polygons)
  - customized for specific applications
  - customized for large data
- Used to answer queries on spatial data such as:
  - point location
  - nearest neighbor (NN)
  - k-NNs
  - range searching
  - find all segments intersecting a given segment
  - meshing
  - …

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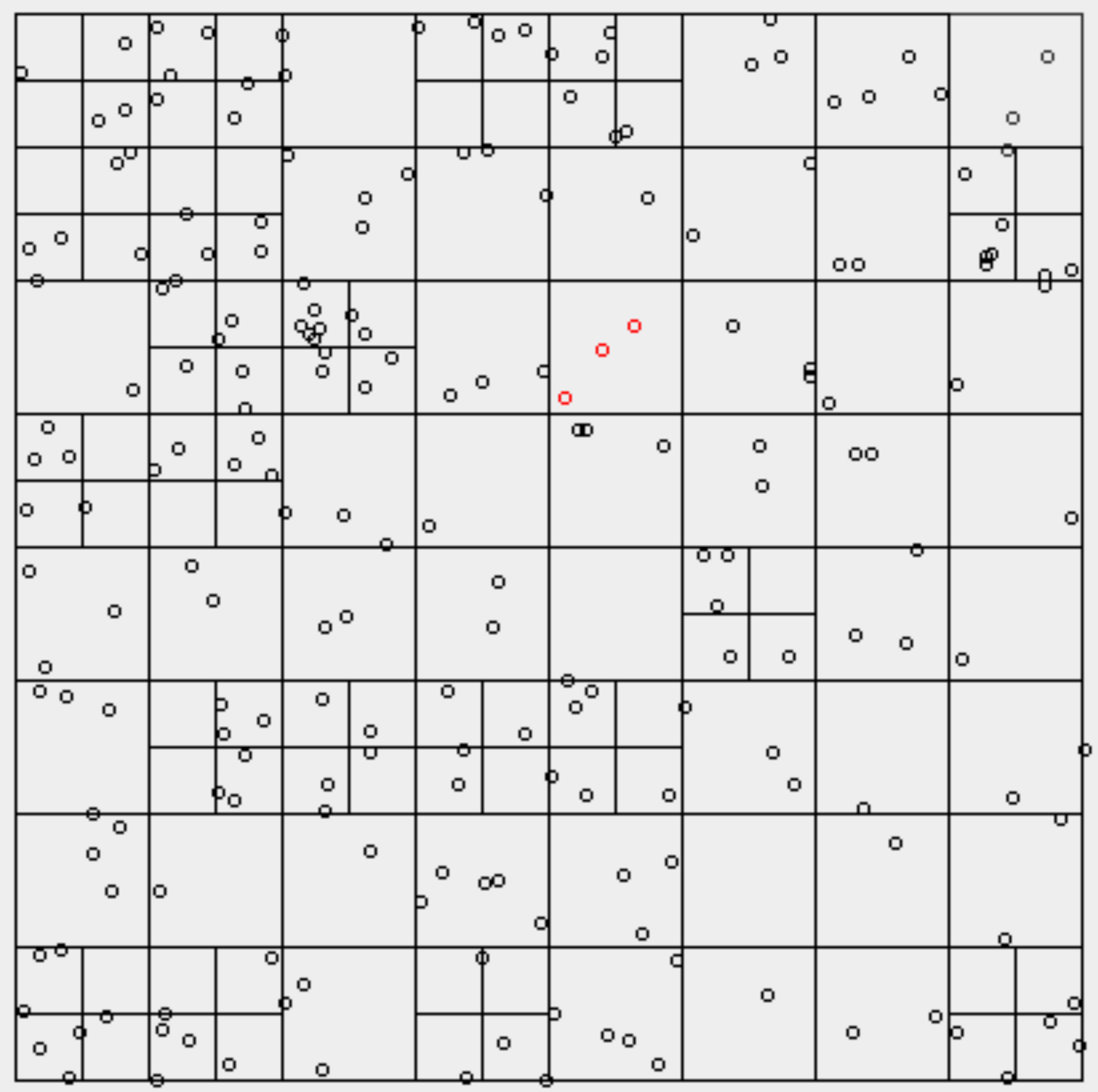

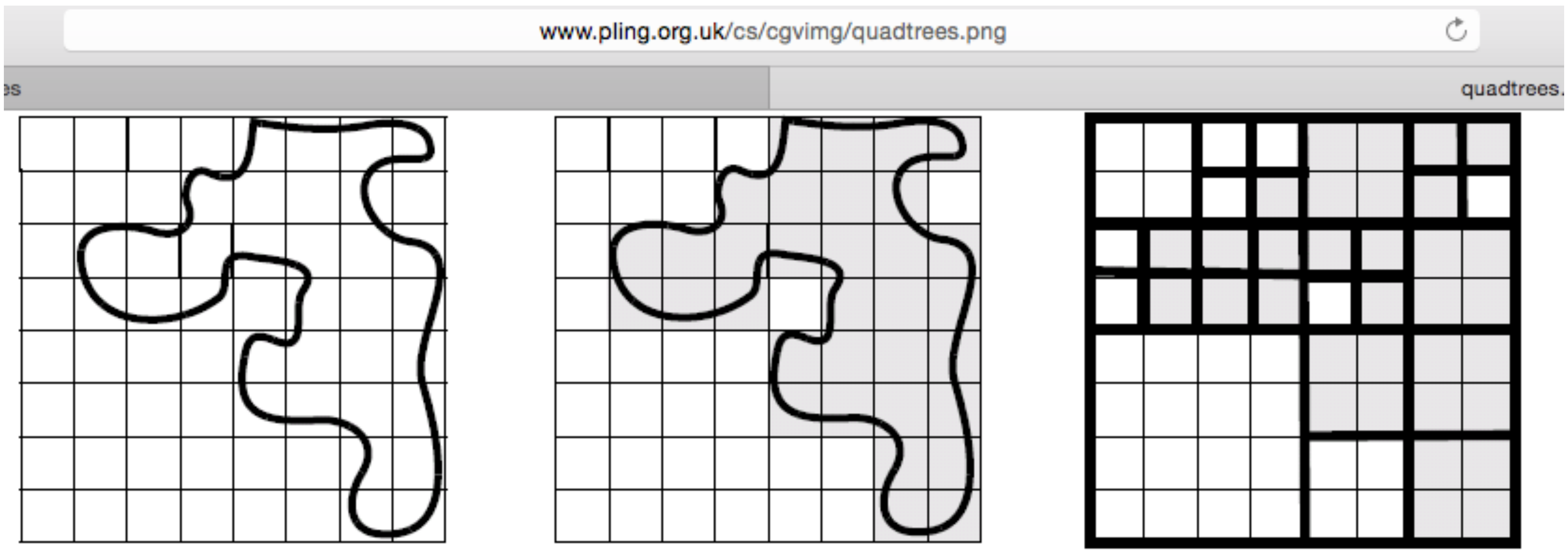
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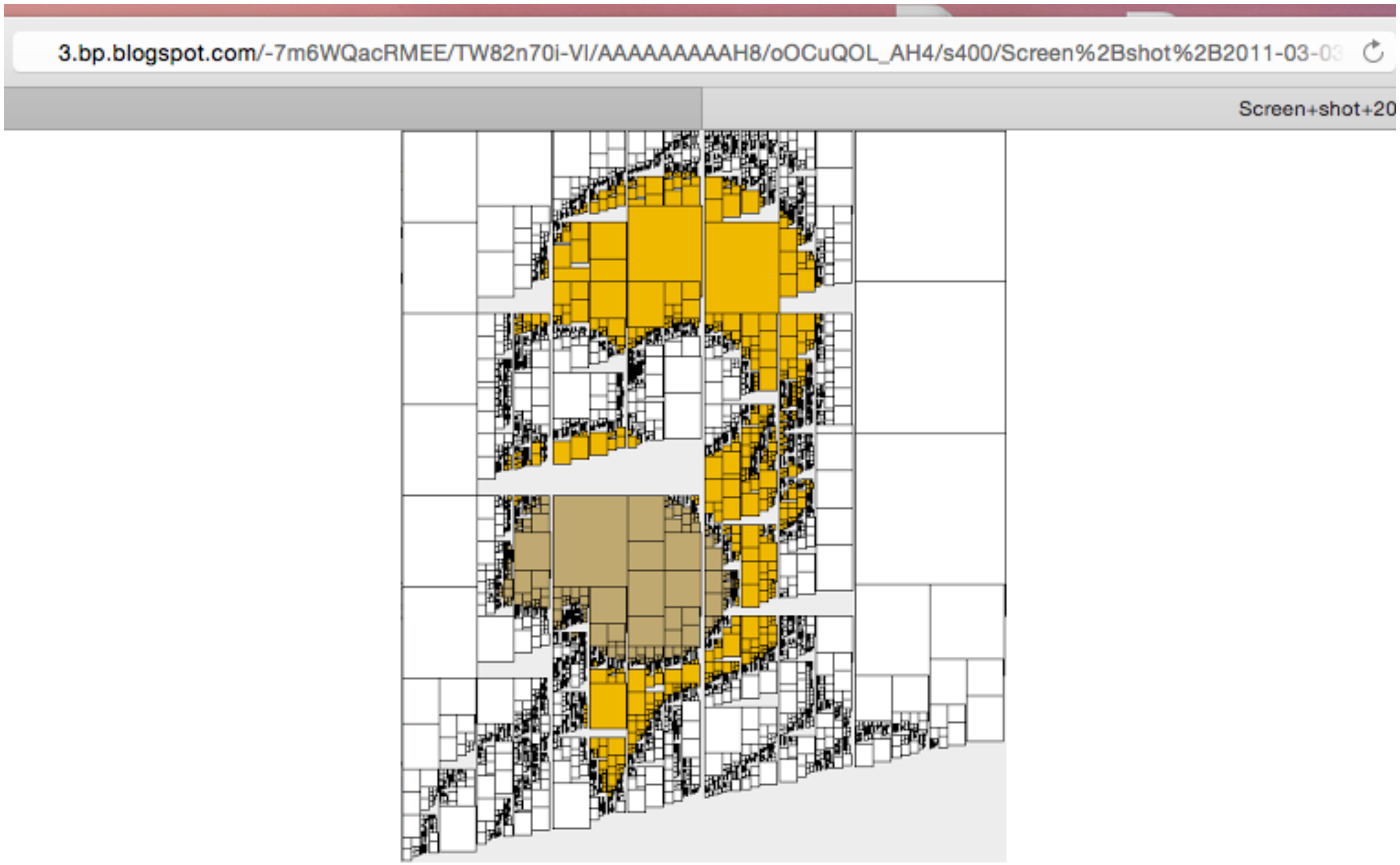
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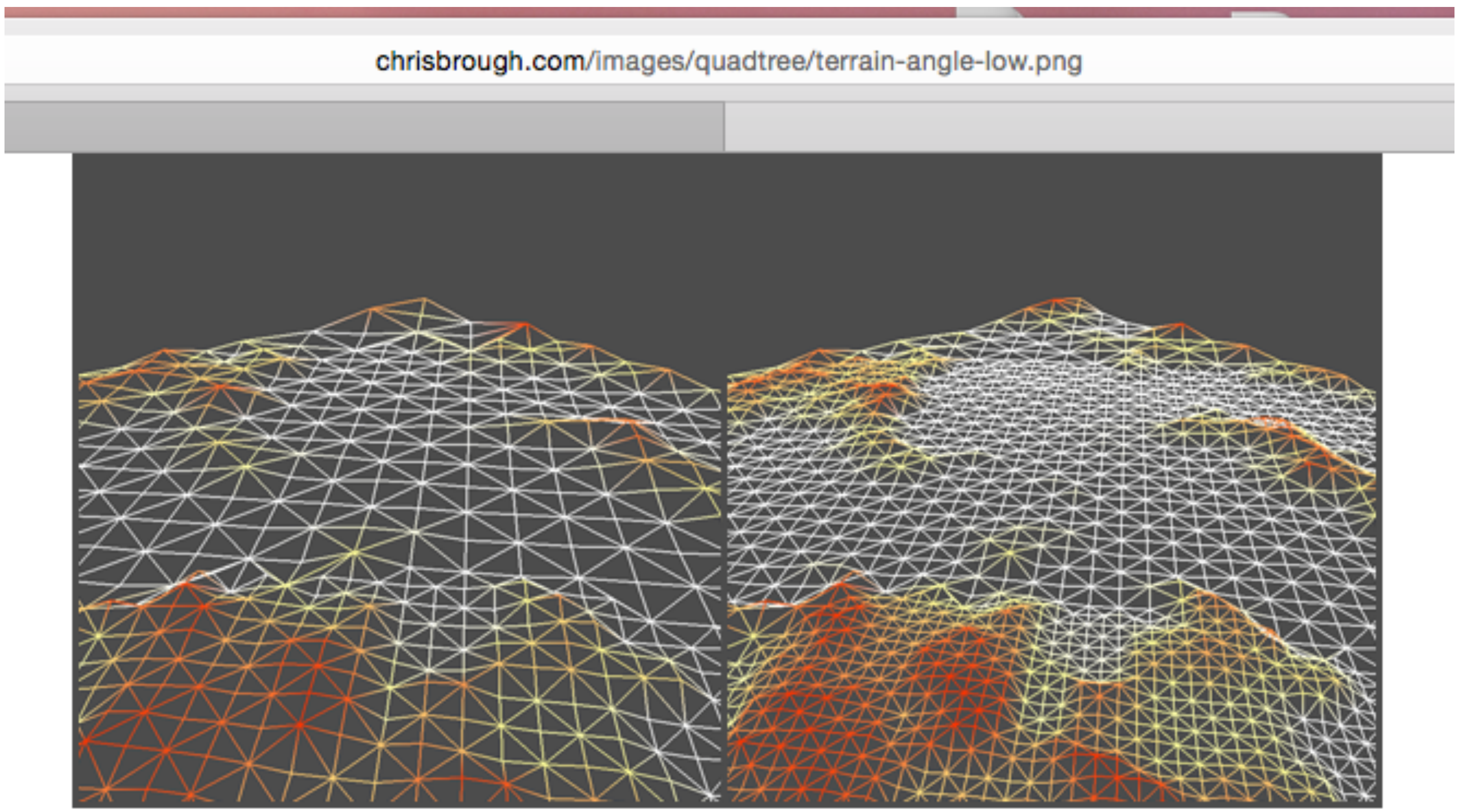
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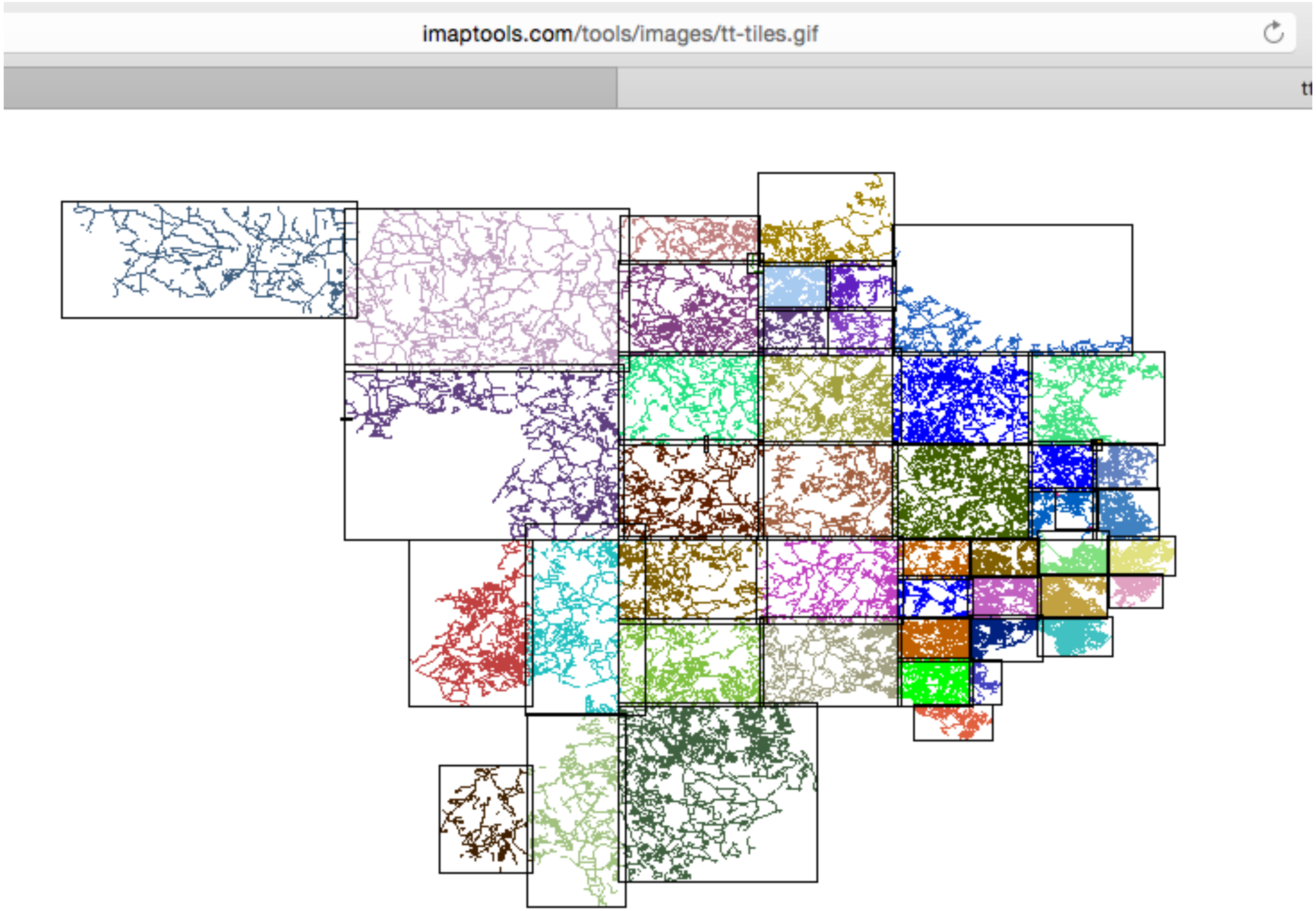
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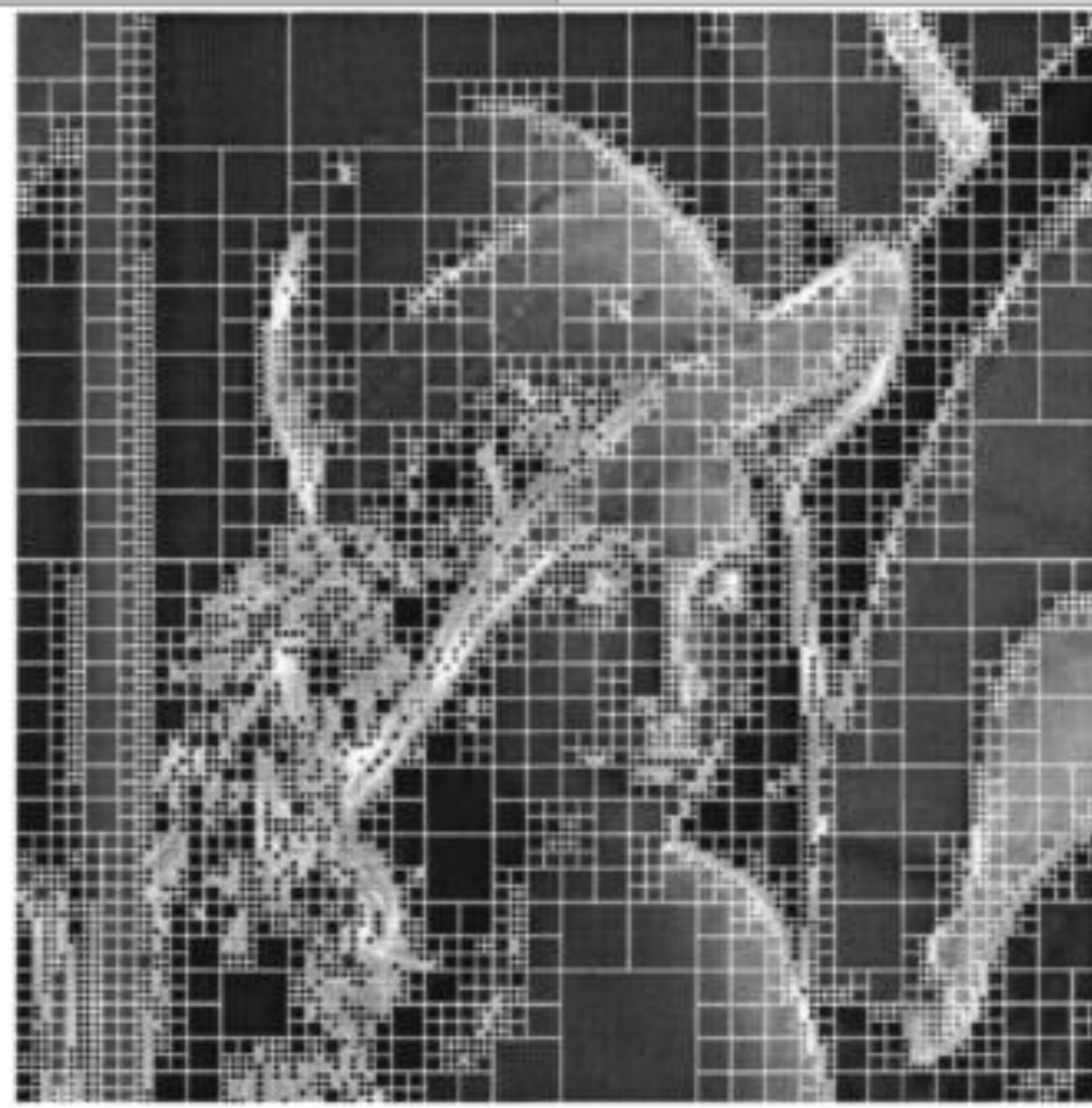

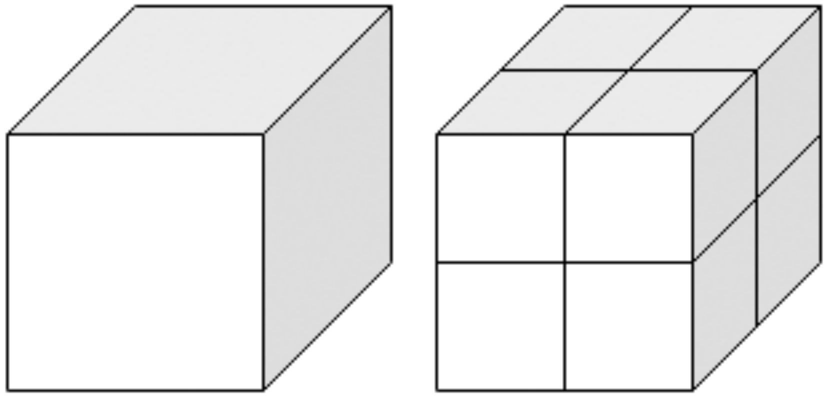

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