Summations

Reading: CLRS A

Why study summations?

1. We saw that a summation came up in the analysis of Insertion-Sort. In general, the running time of a *while* loop can be expressed as the sum of the running time of each iteration.

2. Summations come up in solving recurrences.

There two basic types of summations that come up all the time in analysis:

- arithmetic summations

- geometric summations

Most of the time, the summations that come up in the analysis of algorithms will reduce to one of these two basic types. There is also a third type that occurs occasionaly, the *harmonic* summation. Below we see definitions and examples.

1 Arithmetic series

The arithmetic series is the following: $1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k$. It can be shown that:

$$\boxed{1+2+3+\cdots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \Theta(n^2)}$$

The credit for the first derivation of this series goes to Gauss. There are several ways to prove this. A proof by induction is given in the appendix.

Note: The arithmetic series can be generalized as follows (can also be shown by induction):

$$\boxed{1+2^d+3^d+...+n^d = \sum_{k=1}^{n} k^d = \Theta(n^{d+1})}$$

2 Geometric series

The geometric series is the following: $1 + x + x^2 + \cdots x^n = \sum_{k=0}^{n} x^k$. It can be shown that:

$$\boxed{1 + x + x^2 + \cdots x^n = \sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}}$$

A proof by induction is given in the appendix.

In algorithm analysis we are interested only in the order of growth. Depending on the value of x we get:

$$\boxed{\begin{array}{l} \text{if } x < 1 : \sum_{k=1}^{n} x^k < \sum_{k=1}^{\infty} x^k = \frac{1}{1-x} = \Theta(1) \\ \text{if } x > 1 : \sum_{k=1}^{n} x^k < \sum_{k=1}^{\infty} x^k = \frac{x^n-1}{x-1} = \Theta(x^n) \\ \text{if } x = 1 : \sum_{k=1}^{n} 1 = n \end{array}}$$

Example: $1 + \frac{1}{2} + \ldots \frac{1}{2^n} = \sum_{k=1}^{n} \frac{1}{2^k} = \Theta(1)$

3 Harmonic Series

Occasionaly we may encounter the following series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. This is called the *harmonic* series. It can be shown that

$$\boxed{\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \Theta(\log n)}$$

4 Summations–overview

We have a sum. We want a (tight) bound for it. What can we do?

- express it in terms of basic summations (for which we know the bound) and plug in the bound

All summations that we'll encounter in this class will fall in this category.

Generally speaking, if the summation does not reduce to one of the few basic summations, we can try to:

- guess a bound and prove it by induction
- obtain upper and lower bounds by bounding terms and/or splitting. For more info, refer to the textbook.

5 Appendix

5.1 Arithmetic summation, proof by induction

- Base case: $n = 1 \Rightarrow \sum_{k=1}^{1} 1 = \frac{1(1+1)}{2} = 1$
- Assume it holds for n: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

  Show it holds for $n+1$: $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2} = \frac{1}{2}n^2 + \frac{3}{2}n + 1$

  Proof:

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{1}{2}n^2 + \frac{1}{2}n + n + 1 \\
&= \frac{1}{2}n^2 + \frac{3}{2}n + 1
\end{aligned}$$

5.2 Geometric summation, proof by induction

- Base case: $n = 1 \Rightarrow \sum_{k=0}^{1} x^k = 1 + x$

  $\frac{x^{n+1}-1}{x-1} = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)} = x + 1$

- Assume claim holds for n: $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$

  Show claim holds for $n+1$: $\sum_{k=0}^{n+1} x^k = \frac{x^{n+2}-1}{x-1}$

  Proof:

$$\begin{aligned}
\sum_{k=0}^{n+1} x^k &= \sum_{k=0}^{n} x^k + x^{n+1} \\
&= \frac{x^{n+1}-1}{x-1} + x^{n+1} \\
&= \frac{x^{n+1}-1+x^{n+1}(x-1)}{x-1} \\
&= \frac{x^{n+1}-1+x^{n+2}-x^{n+1}}{x-1} \\
&= \frac{x^{n+2}-1}{x-1}
\end{aligned}$$