Linear Time Selection

(CLRS 9)

The selection problem is the following: Given an array A of n elements and a value i ($1 \leq i \leq n$), find ith smallest element in an array. This element is called the element of $rank = i$.

For simplicity, we assume the elements are distinct.

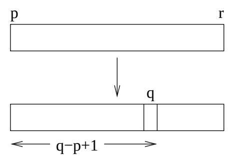
SELECT(A, i): returns the i'th smallest element in A

Note that for $i = 1$ we want the *m*inimum element, and for $i = n$ we want the *maximum* element. The element of rank $i = n/2$ is called the *median*.

- How fast can you find the ith smallest element?
- Straightforward algorithm: Sort and return $A[i]$. $O(n \lg n)$
- Can we do better?
- Let's look at some special cases of SELECT(i)
  - Minimum or maximum can easily be found in $n - 1$ comparisons
    - Scan through elements maintaining minimum/maximum
  - Second largest/smallest element can be found in $(n - 1) + (n - 2) = 2n - 3$ comparisons
    - Find and remove minimum/maximum
    - Find minimum/maximum
  - Median:
    - Using the above idea repeatedly we can find the median in time $\sum_{i=1}^{n/2}(n-i) = n^2/2 - \sum_{i=1}^{n/2} i = n^2/2 - (n/2 \cdot (n/2 + 1))/2 = \Theta(n^2)$
    - We can easily design $\Theta(n \log n)$ algorithm using sorting
- Can we design $O(n)$ time algorithm for general i?
- If we could partition nicely (which is what we are really trying to do) we could solve the problem
  - by partitioning and then recursively looking for the element in one of the partitions:

SELECT(A, p, r, i)

IF $p = r$ THEN RETURN $A[p]$
q=PARTITION(A, p, r)



$k = q - p + 1$
IF $i \leq k$ THEN
    RETURN SELECT(A, p, q, i)
ELSE
    RETURN SELECT$(A, q + 1, r, i - k)$
FI

Select i'th elements using SELECT$(A, 1, n, i)$

– If the partition was perfect ($q = n/2$) we have

$$\begin{aligned}
T(n) &= T(n/2) + n \\
&= n + n/2 + n/4 + n/8 + \cdots + 1 \\
&= \sum_{i=0}^{\log n} \frac{n}{2^i} \\
&= n \cdot \sum_{i=0}^{\log n} (\frac{1}{2})^i \\
&\leq n \cdot \sum_{i=0}^{\infty} (\frac{1}{2})^i \\
&= \Theta(n)
\end{aligned}$$

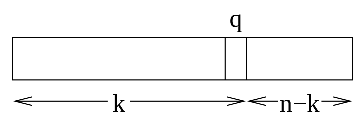
Note:

* The trick is that we only recurse on one side.
* In the worst case the algorithm runs in $T(n) = T(n-1) + n = \Theta(n^2)$ time.
* We could use randomization to get good expected partition.
* Even if we just always partition such that a constant fraction ($\alpha < 1$) of the elements are eliminated we get running time $T(n) = T(\alpha n) + n = n \sum_{i=0}^{\log n} \alpha^i = \Theta(n)$.

- It turns out that we can modify the algorithm and get $T(n) = \Theta(n)$ in the worst case
  - The idea is to find a split element q such that we always eliminate a fraction of the elements:

SELECT(i)

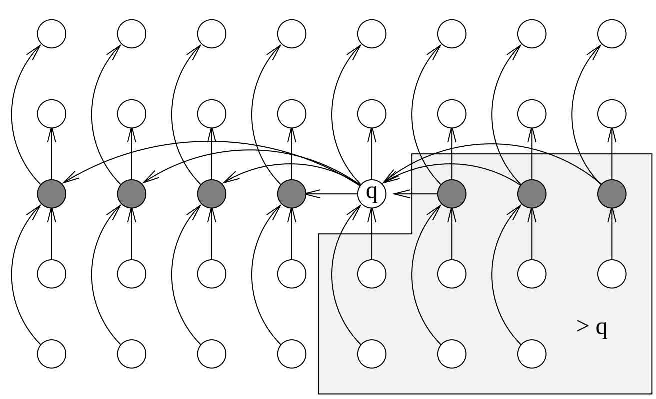
* Divide n elements into groups of 5
* Select median of each group ($\Rightarrow \lceil \frac{n}{5} \rceil$ selected elements)
* Use SELECT recursively to find median q of selected elements
* Partition *all* elements based on q



* Use SELECT recursively to find i'th element
  · If $i \leq k$ then use SELECT(i) on k elements
  · If $i > k$ then use SELECT($i-k$) on $n-k$ elements

  - If n' is the maximal number of elements we recurse on in the last step of the algorithm the running time is given by $T(n) = \Theta(n) + T(\lceil \frac{n}{5} \rceil) + \Theta(n) + T(n')$

- Estimation of n':
  - Consider the following figure of the groups of 5 elements
    * An arrow between element e_1 and e_2 indicates that $e_1 > e_2$
    * The $\lceil \frac{n}{5} \rceil$ selected elements are drawn solid (q is median of these)
    * Elements $> q$ are indicated with box



- Number of elements $> q$ is *larger* than $3(\frac{1}{2}\lceil\frac{n}{5}\rceil - 2) \geq \frac{3n}{10} - 6$
    - We get 3 elements from each of $\frac{1}{2}\lceil\frac{n}{5}\rceil$ columns except possibly the one containing q and the last one.
- Similarly the number of elements $< q$ is *larger* than $\frac{3n}{10} - 6$

  $\Downarrow$

  We recurse on at most $n' = n - (\frac{3n}{10} - 6) = \frac{7}{10}n + 6$ elements

- So SELECTION(i) runs in time $T(n) = \Theta(n) + T(\lceil\frac{n}{5}\rceil) + T(\frac{7}{10}n + 6)$
- Solution to $T(n) = n + T(\lceil\frac{n}{5}\rceil) + T(\frac{7}{10}n + 6)$:
  - Guess $T(n) \leq cn$
  - Induction:

$$\begin{aligned}
T(n) &= n + T(\lceil\frac{n}{5}\rceil) + T(\frac{7}{10}n + 6) \\
&\leq n + c \cdot \lceil\frac{n}{5}\rceil + c \cdot (\frac{7}{10}n + 6) \\
&\leq n + c\frac{n}{5} + c + \frac{7}{10}cn + 6c \\
&= \frac{9}{10}cn + n + 7c \\
&\leq cn
\end{aligned}$$

    If $7c + n \leq \frac{1}{10}cn$ which can be satisfied (e.g. true for $c = 20$ if $n > 140$)
  - Note: It is important that we chose every 5'th element, not all other choices will work (homework) (Note: This algorithm gives $\sim 16n$ comparisons. Best know $\sim 2.95n$. Best lower bound $> 2n$).

Selection and Quicksort

- Recall that the running time of Quicksort depends on how good the partition is
  - Best case ($q = n/2$): $T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n)$.
  - Worst case ($q = 1$): $T(n) = T(1) + T(n-1) + \Theta(n) \Rightarrow T(n) = \Theta(n^2)$.
  - Expected case for randomized algorithm: $\Theta(n \log n)$
- In Quicksort: if we could find element e such that $rank(e) = n/2$ (the *median*) in $O(n)$ time we could make quick-sort run in $\Theta(n \log n)$ time worst case.
  - We could just exchange e with last element in A in beginning of PARTITION and thus make sure that A is always partition in the middle