Summations

Reading: CLRS A

Why study summations?

1. We saw that a summation came up in the analysis of Insertion-Sort. In general, the running time of a *while* loop can be expressed as the sum of the running time of each iteration.

2. Summations come up in solving recurrences.

1 Basic Summations

1.1 Arithmetic series

$\boxed{\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2} = \Theta(n^2)}$

How can we prove this? By induction:

- Base case: $n = 1 \Rightarrow \sum_{k=1}^{1} 1 = \frac{1(1+1)}{2} = 1$

- Assume it holds for n: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

  Show it holds for $n+1$: $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2} = \frac{1}{2}n^2 + \frac{3}{2}n + 1$

  Proof:

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{1}{2}n^2 + \frac{1}{2}n + n + 1 \\
&= \frac{1}{2}n^2 + \frac{3}{2}n + 1
\end{aligned}$$

Note: In general it can be shown by induction that $\boxed{\sum_{k=1}^{n} k^d = \Theta(n^{d+1})}$

1.2 Geometric series

$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots x^n = \frac{x^{n+1}-1}{x-1}$

Proof by induction:

- Base case: $n = 1 \Rightarrow \sum_{k=0}^{1} x^k = 1 + x$
  $\frac{x^{n+1}-1}{x-1} = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)} = x + 1$

- Induction:

  Assume claim holds for n: $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$

  Show claim holds for $n+1$: $\sum_{k=0}^{n+1} x^k = \frac{x^{n+2}-1}{x-1}$

  Proof:

$$\begin{aligned}
\sum_{k=0}^{n+1} x^k &= \sum_{k=0}^{n} x^k + x^{n+1} \\
&= \frac{x^{n+1}-1}{x-1} + x^{n+1} \\
&= \frac{x^{n+1} - 1 + x^{n+1}(x-1)}{x-1} \\
&= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x-1} \\
&= \frac{x^{n+2}-1}{x-1}
\end{aligned}$$

if $x < 1 : \sum_{k=1}^{n} x^k < \sum_{k=1}^{\infty} x^k = \frac{1}{1-x} = \Theta(1)$
if $x > 1 : \sum_{k=1}^{n} x^k < \sum_{k=1}^{\infty} x^k = \frac{x^n-1}{x-1} = \Theta(x^n)$
if $x = 1 : \sum_{k=1}^{n} 1 = n$

Example: $1 + \frac{1}{2} + \ldots \frac{1}{2^n} = \sum_{k=1}^{n} \frac{1}{2^k} = \Theta(1)$

1.3 Harmonic Series

$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \Theta(\log n)$

We will not do a proof for this one.

2 Summations–overview

We have a sum. We want a (tight) bound for it. What can we do?

- express it in terms of basic summations (for which we know the bound)

All summations that we'll encounter in the class will fall in this category. Generally speaking, if the summation does not reduce to one of the few basic summations, we can try to:

- guess a bound and prove it by induction
- obtain upper and lower bounds by bounding terms and/or splitting. For more info, refer to the textbook.