Recurrence example

Consider the following recurrence which is very similar to the one for Mergesort:

$$T(n) = 3T(n/3) + n$$

Below we solve it by iteration and then by substitution.

1 Iteration

$$\begin{aligned}
T(n) &= 3T(n/3) + n \\
&= 3(3T(n/9) + n/3)) + n \\
&= 9T(n/9) + 2n \\
&= 9(3T(n/27 + n/9) + 2n \\
&= 27T(n/27) + 3n \\
&= \ldots \\
&= 3^i T(n/3^i) + i \cdot n
\end{aligned}$$

- Recursion depth: How long (how many iterations) it takes until the subproblem has constant size? i iterations, where $\frac{n}{3^i} = 1 \Rightarrow i = \log_3 n$
- What is the last term? $3^i T(1) = 3^{\log_3 n} T(1) = n \cdot 1 = n$

  So we have:

$$\begin{aligned}
T(n) &= 3^{\log_3 n} T(1) + \log_3 n \\
&= n + \log_3 n \cdot n \\
&= \Theta(n \lg n)
\end{aligned}$$

2 Substitution

With substitution we need to "guess" the result and prove it by induction. Unfortunately there is no easy recipe on how to "guess" —- just through exercises, by building intuition. We usually start a recurrence by iteration, get a feel for the answer, and then formally prove the result with induction.

Guess $T(n) \leq cn \log_3 n$ for some constant c, for any $n \geq n_0$.

Proof:

- Base case: we need to show that our guess holds for some base case (not necessarily $n = 1$, some small n is ok).

  Let's try for $n = 1$: $T(1) = 1 < c \cdot 1 \cdot \log_3 1$ this is not true.

  let's try $n = 3$: $T(3) = 3T(1) + 3 = 6 < c3 \log_3 3 = 3c$, true for $c \geq 2$.

- Inductive hypothesis: Assume our guess/claim holds for $n/3$, that is, $T(n/3) \leq c\frac{n}{3} \log_3 \frac{n}{3}$

  Prove that this implies that it holds for n: $T(n) \leq cn \log_3 n$

  Proof:

$$
\begin{aligned}
T(n) &= 3T(n/3) + n \\
&\leq 3(c\frac{n}{3} \log_3 \frac{n}{3}) + n (\text{by inductive hyp}) \\
&= cn \log_3 \frac{n}{3} + n \\
&= cn \log_3 n - cn \log_3 3 + n \\
&= cn \log_3 n - cn + n
\end{aligned}
$$

  We want to prove that $T(n) \leq cn \log_3 n$, and this is true when $c \geq 1$

Putting together both conditions on c (one from the base case, and one from the induction hypothesis), we get $c \geq 2$. So we can chose $c = 2$ and the claim is true. Thus we proved that $T(n) \leq 2n \log_3 n$.

3 Other recurrences

Some important/typical recurrences:

- Logarithmic: $\Theta(\log n)$
  - Recurrence: $T(n) = 1 + T(n/2)$
  - Typical example: Recurse on half the input (and throw half away)
  - Variations: $T(n) = 1 + T(99n/100)$
- Linear: $\Theta(N)$
  - Recurrence: $T(n) = 1 + T(n - 1)$
  - Typical example: Single loop
  - Variations:
    * $T(n) = 1 + 2T(n/2)$
    * $T(n) = n + T(n/2)$
    * $T(n) = T(n/5) + T(7n/10 + 6) + n$
- Quadratic: $\Theta(n^2)$
  - Recurrence: $T(n) = n + T(n - 1)$
  - Typical example: Nested loops
- Exponential: $\Theta(2^n)$
  - Recurrence: $T(n) = 2T(n - 1)$