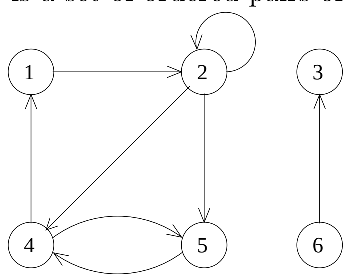
Basic Graph Algorithms

(CLRS B.4-B.5, 22.1-22.4)

1 Basic Graph Definitions

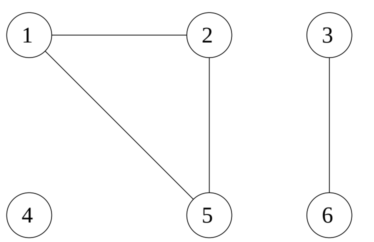
- A graph $G = (V, E)$ consists of a finite set of *vertices* V and a finite set of *edges* E.
  - *Directed graphs*: E is a set of ordered pairs of vertices (u, v) where $u, v \in V$



V = {1, 2, 3, 4, 5, 6}

E = {(1,2), (2,2), (2,4), (2,5)
(4,1), (4,5), (5,4), (6,3)

  - *Undirected graph*: E is a set of unordered pairs of vertices $\{u, v\}$ where $u, v \in V$



V = {1, 2, 3, 4, 5, 6}

E = {{1,2}, {1,5}, {2,5}, {3,6}}

- Edge (u, v) is *incident* to u and v
- *Degree* of vertex in undirected graph is the number of edges incident to it.
- *In (out) degree* of a vertex in directed graph is the number of edges entering (leaving) it.
- A *path* from u_1 to u_2 is a sequence of vertices $< u_1{=}v_0, v_1, v_2, \cdots, v_k{=}u_2 >$ such that $(v_i, v_{i+1}) \in E$ (or $\{v_i, v_{i+1}\} \in E$)
  - We say that u_2 is *reachable* from u_1
  - The *length* of the path is k
  - It is a *cycle* if $v_0 = v_k$
- An undirected graph is *connected* if every pair of vertices are connected by a path
  - The *connected components* are the equivalence classes of the vertices under the "reachability" relation. (All connected pair of vertices are in the same connected component).

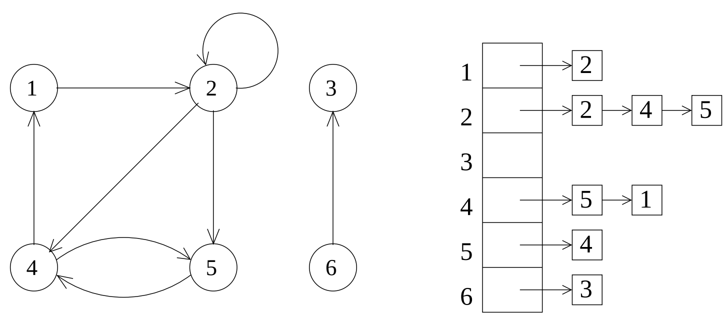
- A directed graph is *strongly connected* if every pair of vertices are reachable from each other
  - The *strongly connected components* are the equivalence classes of the vertices under the "mutual reachability" relation.

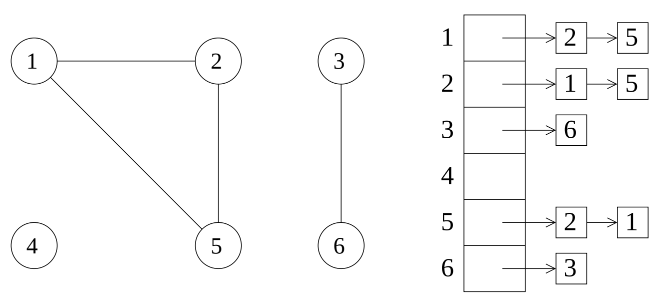
- Graphs appear all over the place in all kinds of applications, e.g:
  - Trees ($|E| = |V| - 1$)
  - Connectivity/dependencies (house building plans, WWW-page connections = internet graph)
- Often the edges (u, v) in a graph have weights $w(u, v)$, e.g.
  - Road networks (distances)
  - Cable networks (capacity)

1.1 Representation

- *Adjacency-list* representation:
  - Array of $|V|$ list of edges incident to each vertex.

    Examples:



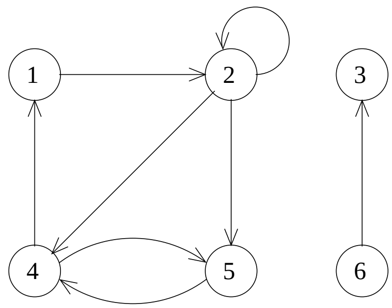


  - Note: For undirected graphs, every edge is stored twice.
  - If graph is weighted, a weight is stored with each edge.
- *Adjacency-matrix* representation:

– $|V| \times |V|$ matrix A where

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Examples:



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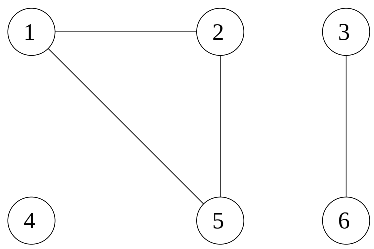
i	1	2	3	4	5	6
1	0	1	0	0	0	0
2	0	1	0	1	1	0
3	0	0	0	0	0	0
4	1	0	0	0	1	0
5	0	0	0	1	0	0
6	0	0	1	0	0	0



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i	1	2	3	4	5	6
1	0	1	0	0	1	0
2	1	0	0	0	1	0
3	0	0	0	0	0	1
4	0	0	0	0	0	0
5	1	1	0	0	0	0
6	0	0	1	0	0	0

– Note: For undirected graphs, the adjacency matrix is symmetric along the main diagonal ($A^T = A$).

– If graph is weighted, weights are stored instead of one's.

- Comparison of matrix and list representation:

Adjacency list	Adjacency matrix
$O(\lvert V\rvert + \lvert E\rvert)$ space	$O(\lvert V\rvert^2)$ space
Good if graph *sparse* ($\lvert E\rvert << \lvert V\rvert^2$)	Good if graph *dense* ($\lvert E\rvert \approx \lvert V\rvert^2$)
No quick access to (u, v)	$O(1)$ access to (u, v)

- We will use adjacency list representation unless stated otherwise ($O(|V| + |E|)$ space).

2 Graph traversal

- There are two standard (and simple) ways of traversing all vertices/edges in a graph in a systematic way
  - Breadth-first
  - Depth-first
- We can use them in many fundamental algorithms, e.g finding cycles, connected components, ...

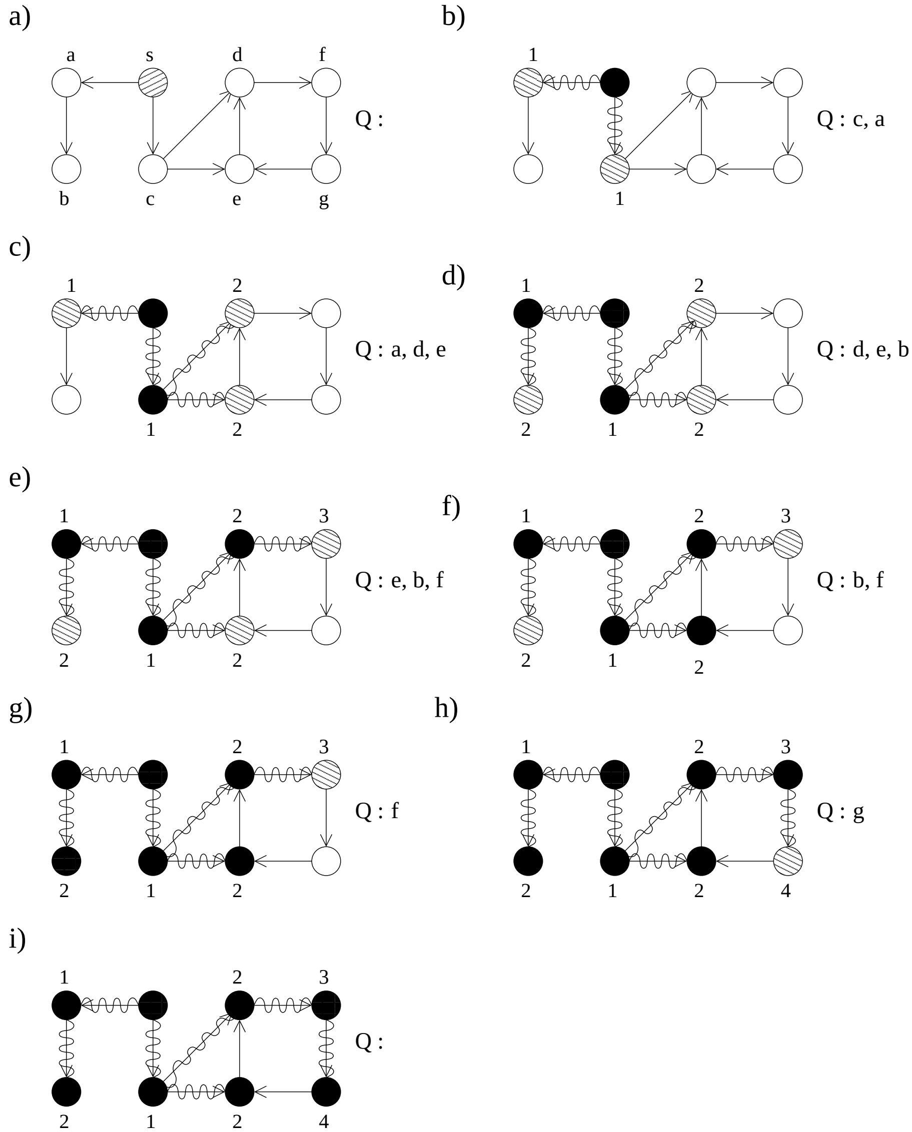
2.1 Breadth-first search (BFS)

- Main idea:
  - Start at some source vertex s and visit,
  - All vertices at distance 1,
  - Followed by all vertices at distance 2,
  - Followed by all vertices at distance 3,

    $\vdots$
- BFS corresponds to computing *shortest path* distance (number of edges) from s to all other vertices.
- To control progress of our BFS algorithm, we think about *coloring* each vertex
  - *White* before we start,
  - *Gray* after we visit the vertex but before we have visited all its adjacent vertices,
  - *Black* after we have visited the vertex and all its adjacent vertices (all adjacent vertices are gray).
- We use a queue Q to hold all gray vertices—vertices we have seen but are still not done with.
- We remember from which vertex a given vertex v is colored gray – i.e. the node that discovered v first; this is called parent[v].
- Algorithm:

```
BFS(s)
    color[s] = gray
    d[s] = 0
    ENQUEUE(Q, s)
    WHILE Q not empty DO
        DEQUEUE(Q, u)
        FOR (u, v) ∈ E DO
            IF color[v] = white THEN
                color[v] = gray
                d[v] = d[u] + 1
                parent[v] = u
                ENQUEUE(Q, v)
            FI
            color[u] = black
    OD
```

- Algorithm runs in $O(|V| + |E|)$ time
- Example (for directed graph):



- Note:
  - parent$[v]$ forms a tree; *BFS-tree*.
  - $d[v]$ contains length of shortest path from s to v. (Prove by induction)
  - We can use parent$[v]$ to find the shortest path from s to a given vertex.
- If graph is not connected we have to try to start the traversal at all nodes.

```
FOR each vertex u ∈ V DO
      IF color[u] = white THEN BFS(u)
OD
```

- Note: We can use algorithm to compute connected components in $O(|V| + |E|)$ time.

2.2 Depth-first search (DFS)

- If we use stack instead of queue Q we get another traversal order; depth-first
  - We go "as deep as possible",
  - Go back until we find unexplored adjacent vertex,
  - Go as deep as possible,

    ⋮
- Often we are interested in "start time" and "finish time" of vertex u
  - *Start time* (d[u]): indicates at what "time" vertex is first visited.
  - *Finish time* (f[u]): indicates at what "time" all adjacent vertices have been visited.
- We can write DFS iteratively using the same algorithm as for BFS but with a STACK instead of a QUEUE, or, we can write a recursive DFS procedure
  - We will color a vertex gray when we first meet it and black when we finish processing all adjacent vertices.
- Algorithm:

```
DFS(u)
    color[u] = gray
    d[u] = time
    time = time + 1
    FOR (u, v) ∈ E DO
        IF color[v] = white THEN
            parent[v] = u
            DFS(v)
        FI
    OD
    color[u] = black
    f[u] = time
    time = time + 1
```

- Algorithm runs in $O(|V| + |E|)$ time
  - As before we can extend algorithm to unconnected graphs and we can use it to detect cycles in $O(|V| + |E|)$ time.

- Example:

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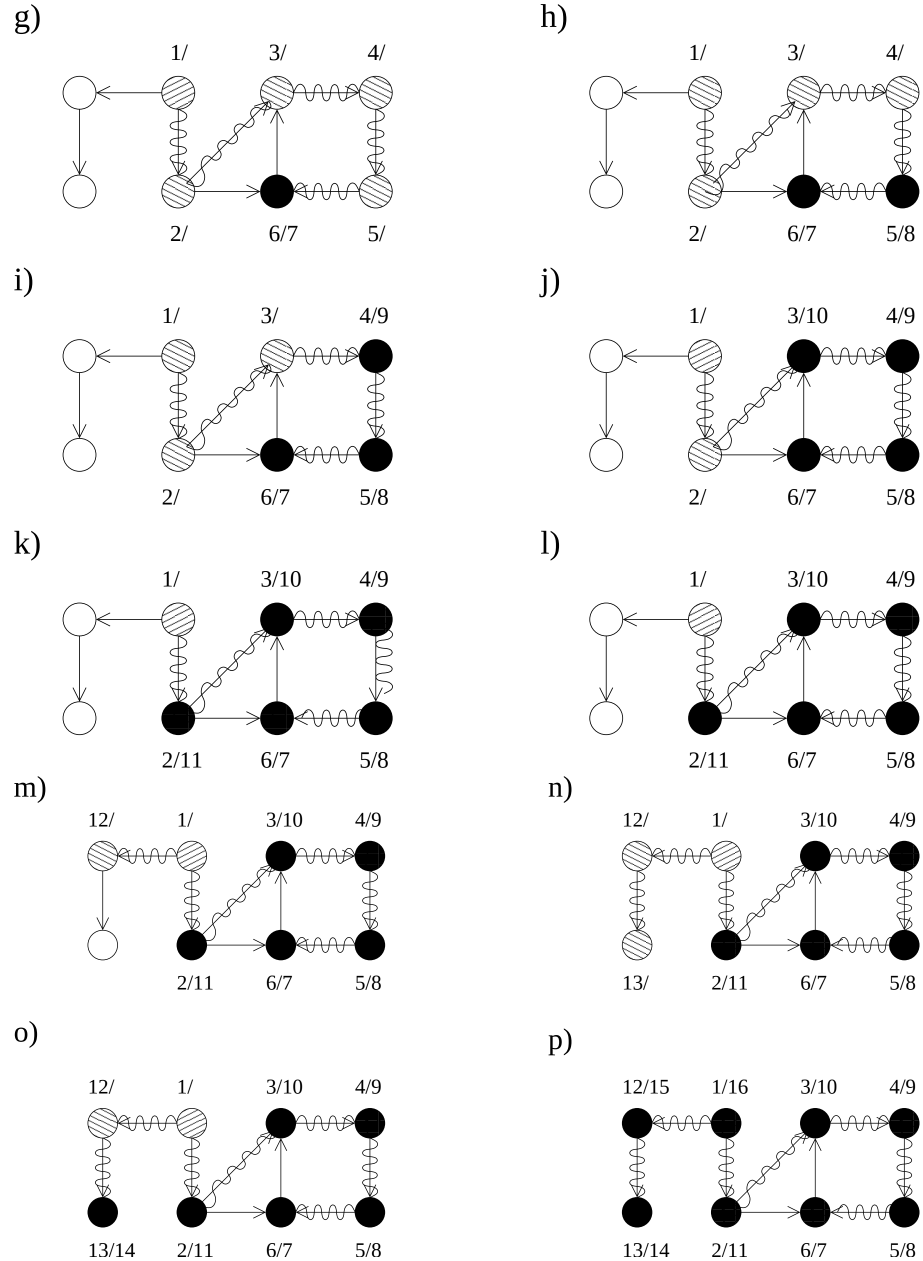
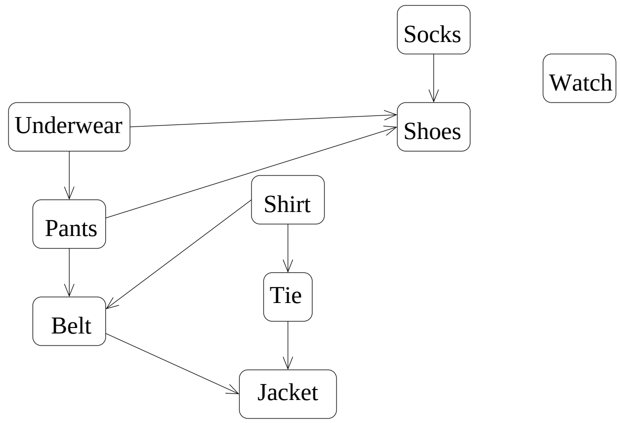


- As previously parent$[v]$ forms a tree; *DFS-tree*
  - Note: If u is descendent of v in DFS-tree then $d[v] < d[u] < f[u] < f[v]$

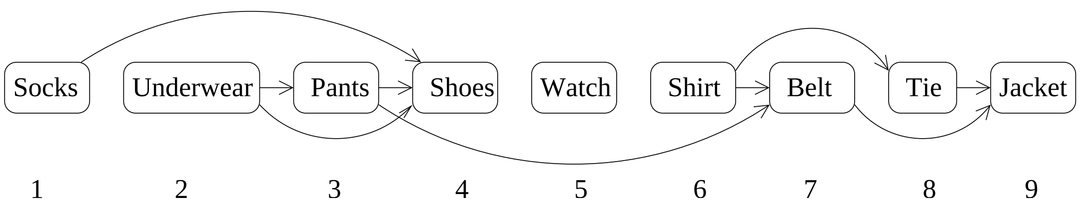
3 Topological sorting

- Definition: Topological sorting of *directed acyclic graph* $G = (V, E)$ is a linear ordering of vertices V such that $(u, v) \in E \Rightarrow u$ appear before v in ordering.

- Topological ordering can be used in scheduling:
  - Example: Dressing (arrow implies "must come before")



We want to compute order in which to get dressed. One possibility:



The given order is one possible topological order.

- Algorithm: Topological order just reverse DFS finish time ($\Rightarrow O(|V| + |E|)$ running time).
- Correctness: $(u, v) \in E \Leftrightarrow f(v) < f(u)$
  - Proof: When (u, v) is explored by DFS algorithm, v must be white or black (gray $\Rightarrow$ cycle).
    * v white: v visited and finished before u is finished $\Rightarrow f(v) < f(u)$
    * v black: v already finished $\Rightarrow f(v) < f(u)$
- Alternative algorithm: Count in-degree of each vertex and repeatedly number and remove in-degree 0 vertex and its outgoing edges: Homework.