Minimum Spanning Trees

(CLRS 23)

- Problem: Given connected, undirected graph $G = (V, E)$ where each edge (u, v) has weight $w(u, v)$. Find acyclic set $T \subseteq E$ connecting all vertices in V with minimal weight $w(T) = \sum_{(u,v) \in T} w(u, v)$.
- An acyclic set connecting all vertices is called a *spanning tree*. We want to find a spanning tree of *minimal weight*. We use *minimum spanning tree* as short for *minimum weight spanning tree*).
- MST problem has many applications
  - For example, think about connecting cities with minimal amount of wire or roads (cities are vertices, weight of edges are distances between city pairs).
- Example:

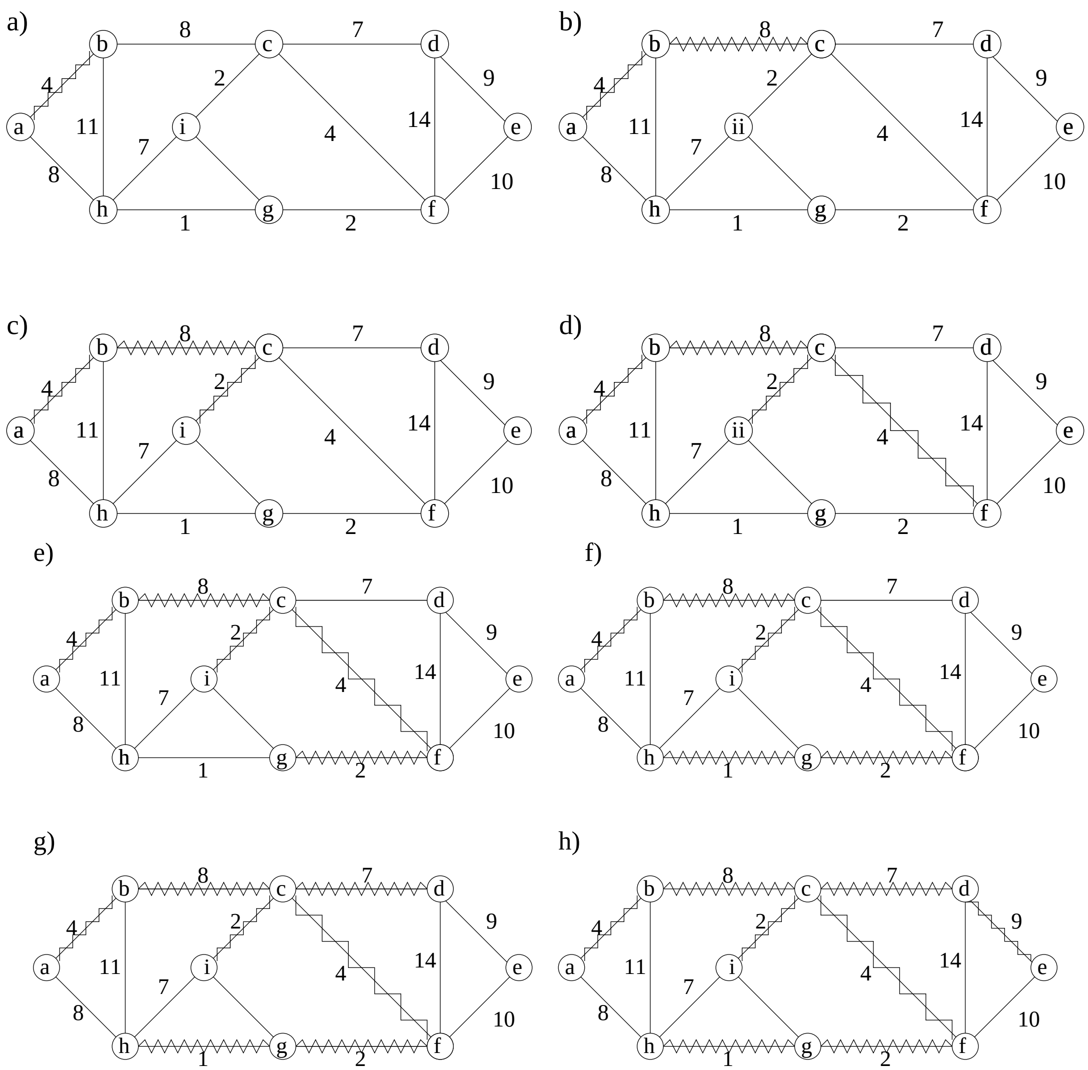
1 PRIM's algorithm

- *Greedy* algorithm for computing MST:
  - Start with spanning tree containing arbitrary vertex r and no edges
  - Grow spanning tree by repeatedly adding minimal weight edge connecting vertex in current spanning tree with a vertex not in the tree
- Implementation:
  - To find minimal edge connected to current tree we maintain a priority queue on vertices not in the tree. The key/priority of a vertex is the weight of minimal weight edge connecting it to the tree. (We maintain pointer from adjacency list entry of v to v in the priority queue).
  - For each node u maintain $visit(u)$ ($(u, visit(u))$ is the cuurently best edge connecting it to the tree.)

```
PRIM(r)

For each v ∈ V DO
    INSERT(PQ, v, ∞)
DECREASE-KEY(PQ, r, 0)
WHILE PQ not empty DO
    u = DELETEMIN(PQ)
    (output edge (u, visit(u)) as part of MST)
    For each (u, v) ∈ E DO
        IF v ∈ PQ and w(u, v) < key(v) THEN
            visit[v] = u
            DECREASE-KEY(PQ, v, w(u, v))
```

– On the example graph, the greedy algorithm would work as follows (starting at vertex a):

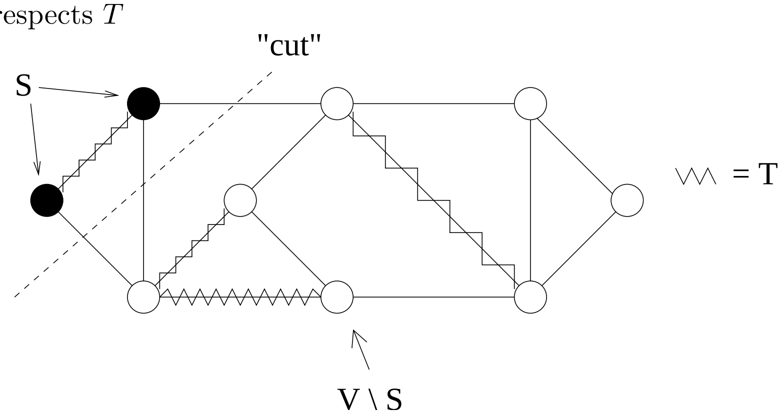


– Analysis:

  * While loop runs $|V|$ times $\Rightarrow$ we perform $|V|$ DELETEMIN's
  * We perform at most one DECREASE-KEY for each of the $|E|$ edges
    $\Downarrow$
    $O((|V| + |E|) \log |V|) = O(|E| \log |V|)$ running time.

– Correctness:

  * When designing a greedy algorithm the hard part is to prove that it works correctly.
  * We will prove a Theorem that allows us to prove the correctness of a general class of greedy MST algorithms:
    Some definitions
    · A *cut* $(S, V \setminus S)$ is a partition of V into sets S and $V \setminus S$
    · A *edge* (u, v) *crosses a cut* S if $u \in S$ and $v \in V \setminus S$ *or* $v \in S$ and $u \in V \setminus S$
    · A *cut* S *respects a set* $T \subseteq E$ if no edge in T crosses the cut
    Example: Cut S respects T

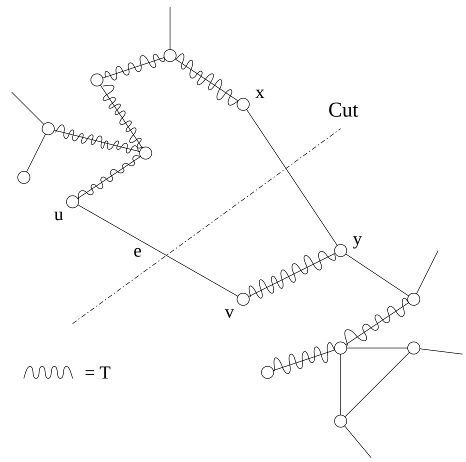


– *Theorem*: If $G = (V, E)$ is a graph such that $T \subseteq E$ is subset of some MST of G, and S is a cut respecting T **then** there is a MST for G containing T and the minimum weight edge $e = (u, v)$ crossing S.

– Note: Correctness of Prim's algorithm follows from the Theorem by induction—cut consist of current spanning tree.

– Proof:

  * Let T^* be MST containing T
  * If $e \in T^*$ we are done
  * If $e \notin T^*$:
    · There must be (at least) one other edge $(x, y) \in T^*$ crossing the cut S such that there is a unique path from u to v in T^* (T^* is spanning tree)



- · This path together with e forms a cycle
- · If we remove edge (x, y) from T^* and add e instead, we still have spanning tree
- · New spanning tree must have same weight as T^* since $w(u, v) \leq w(x, y)$

  $\Downarrow$

  There is a MST containing T and e.

– The Theorem allows us to describe a very abstract greedy algorithm for MST:

```
T = ∅
While |T| ≤ |V| − 1 DO
    Find cut S respecting T
    Find minimal edge e crossing S
    T = T ∪ {e}
```

- ∗ Prim's algorithm follows this abstract algorithm.
- ∗ Kruskal's algorithm is another implementation of the abstract algorithm.

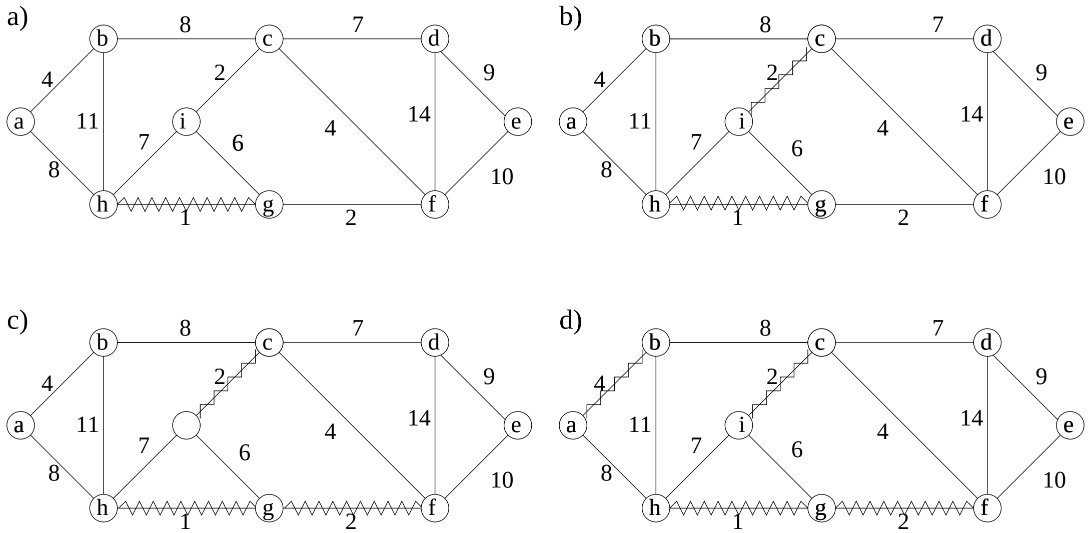
2 Kruskal's Algorithm

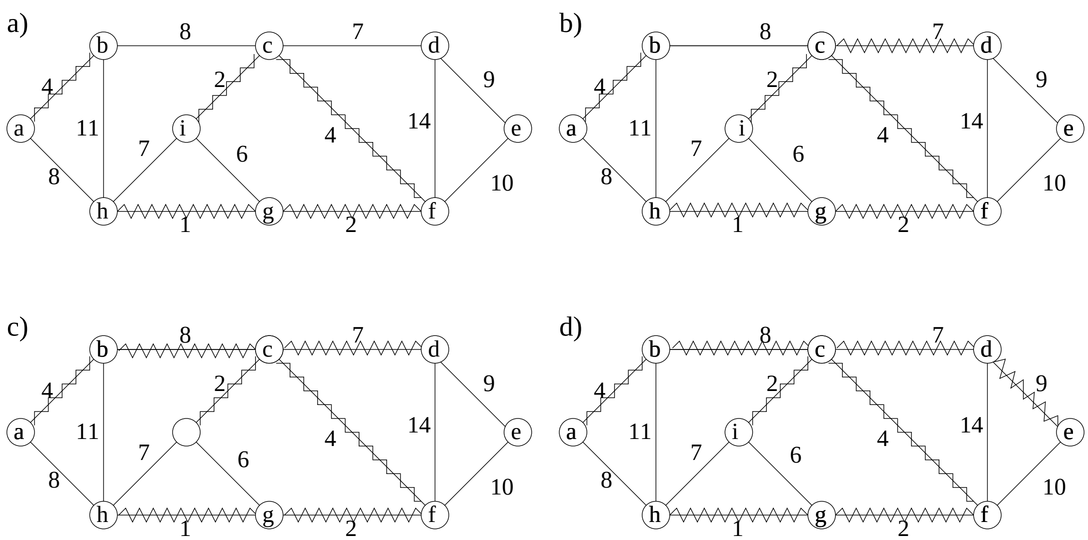
– Kruskal's algorithm is another implementation of the abstract algorithm.

– Idea in Kruskal's algorithm:

  - ∗ Start with $|V|$ trees (one for each vertex)
  - ∗ Consider edges E in increasing order; add edge if it connects two trees

– Example:





– Implementation:

We need (Union-Find) data structure that supports:

- ∗ MAKE-SET(v): Create set consisting of v
- ∗ UNION-SET(u, v): Unite set containing u and set containing v
- ∗ FIND-SET(u): Return unique representative for set containing u

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KRUSKAL

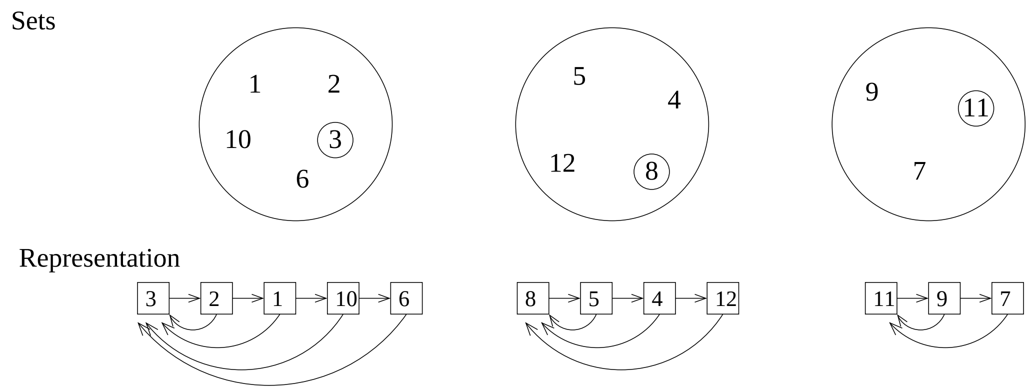
T = ∅
FOR each vertex v ∈ V MAKE-SET(v)
Sort edges of E in increasing order by weight
FOR each edge e = (u, v) ∈ E in order DO
    IF FIND-SET(u) ≠ FIND-SET(v) THEN
        T = T ∪ {e}
        UNION-SET(u, v)
```

- Analysis:
  - We use $O(|E| \log |E|)$ time to sort edges and we perform $|V|$ MAKE-SET, $|V| - 1$ UNION-SET, and $2|E|$ FIND-SET operations.
  - We will discuss a simple solution to the *Union-Find problem* such that MAKE-SET and FIND-SET take $O(1)$ time and UNION-SET takes $O(\log V)$ time amortized.
    $\Downarrow$
    Kruskal's algorithm runs in time $O(|E| \log |E| + |V| \log |V|) = O((|E| + |V|) \log |E|) = O(|E| \log |V|)$ like Prim's algorithm.
- Correctness
  - follows from Theorem above: If minimal edge connects two trees then there exists a cut respecting the current set of edges (cut consisting of vertices in one of the trees)

3 Union-Find

- The *Union-Find problem*: Maintain a set system under:
  - MAKE-SET(v): Create set consisting of v
  - UNION-SET(u, v): Unite set containing u and set containing v
  - FIND-SET(u): Return unique representative for set containing u
- Simple solution:
  - Maintain elements in same set as a linked list with each element having a pointer to the first element in the list (unique representative)

    Example:



  * Make-Set(v): Make a list with one element $\Rightarrow O(1)$ time
  * Find-Set(u): Follow pointer and return unique representative $\Rightarrow O(1)$ time
  * Union-Set(u, v): Link first element in list with unique representative Find-Set(u) after last element in list with unique representative Find-set(v) $\Rightarrow O(|V|)$ time (as we have to update all unique representative pointers in list containing u)

- With this simple solution the $|V| - 1$ Union-Set operations in Kruskal's algorithm may take $O(|V|^2)$ time.
- We can improve the performance of Union-Set with a very simple modification: Always link the smaller list after the longer list ($\Rightarrow$ update the pointers of the smaller list)
  * One Union-Set operation can still take $O(|V|)$ time, but the $|V| - 1$ Union-Set operations takes $O(|V| \log |V|)$ time altogether (one Union-Set takes $O(\log |V|)$ time *amortized*):
    · Total time is proportional to number of unique representative pointer changes
    · Consider element u:
    After pointer for u is updated, u belongs to a list of size at least double the size of the list it was in before
    $\Downarrow$
    After k pointer changes, u is in list of size at least 2^k
    $\Downarrow$
    Pointer can be changed at most $\log |V|$ times.
- With improvement, Kruskal's algorithm runs in time $O(|E| \log |E| + |V| \log |V|) = O((|E| + |V|) \log |E|) = O(|E| \log |V|)$ like Prim's algorithm.