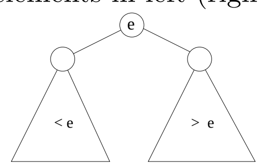
Augmented Search Trees

(CLRS 14)

1 Red-Black Trees

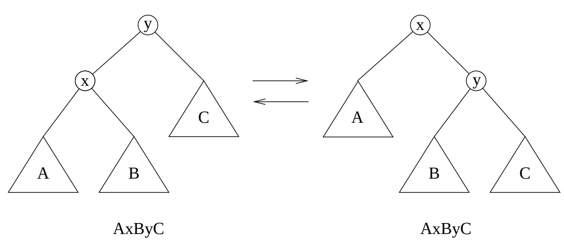
- Last time we discussed red-black trees:
  - Balanced binary trees—all elements in left (right) subtree of node x are $< x$ ($> x$).



  - Every node is colored Red or Black and we maintained red-blue invariant:

∗ Root is Black. ∗ A Red node can only have Black children. ∗ Every path from the root to a leaf contains the same number of Black nodes.

- We saw how the red-blue invariant guaranteed $O(\log n)$ height.
- We could reestablish the red-blue invariant after an insertion or deletion in $O(\log n)$ time
  - $O(\log n)$ node recolorings (no structural changes).
  - $O(1)$ rotations:

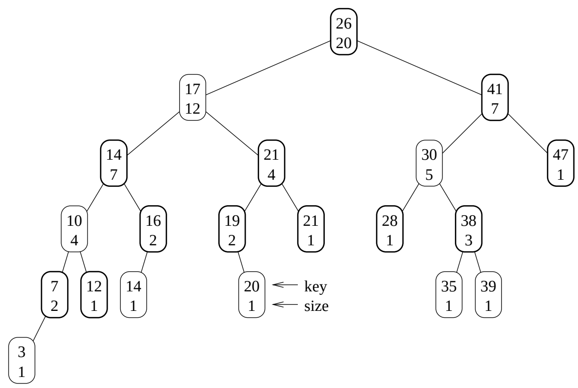


- Red-black tree also supports Search, Successor, and Predecessor in $O(\log n)$ as in binary search trees.
- We will now discuss how to develop data structures supporting other operations by *augmenting* red-black tree.

2 Augmented Data Structures

- We want to add an operation SELECT(i) to a red-black tree
  - We have previously seen how to select the i'th element among n elements in $O(n)$ time.
  - Can we support it faster if we have the elements stored in a data structure?
  - We can of course support the operation in $O(1)$ time if we have the elements sorted in an array but what is we also want to be able to insert and delete elements?
- We augment every node x in red-black tree with a field $size(x)$ equal to the number of nodes in the subtree rooted in x
  - $size(x) = size(left(x)) + size(right(x)) + 1$

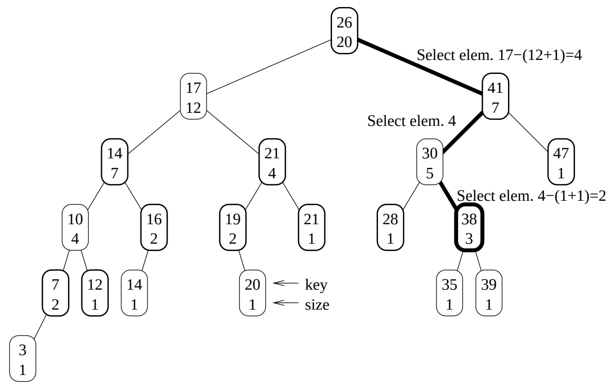
  Example:



- We can use this field to implement SELECT(i):

> SELECT(x, i)
>
> $r = size(left(x)) + 1$
>
> IF $i = r$ THEN Return x
>
> IF $i < r$ THEN Return Select$(left(x), i)$
>
> IF $i > r$ THEN Return Select$(right(x), i-r)$

Example (SELECT(17)):



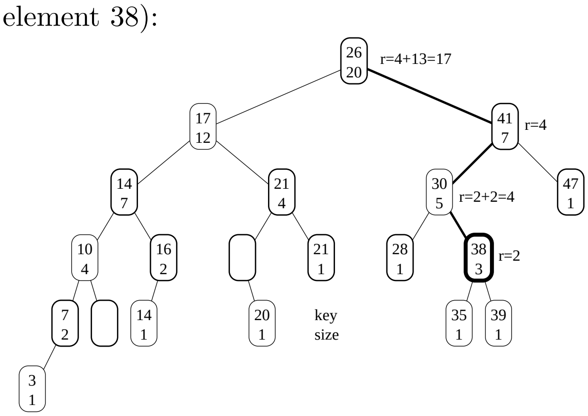
⇓

Since we only follow one root-leaf path the operation takes $O(\log n)$ time.

- Actually, we can also use the field to perform the "opposite" operation in $O(\log n)$ time—determining the *rank* of the element in node x:

> RANK(x)
>
> $r = size(left(x)) + 1$
>
> $y = x$
>
> WHILE $y \neq$ root of tree DO
>
> IF $y = right(parent(y))$ THEN
>
> $r = r + size(left(parent(y))) + 1$
>
> FI
>
> $y = parent(y)$
>
> OD
>
> Return r

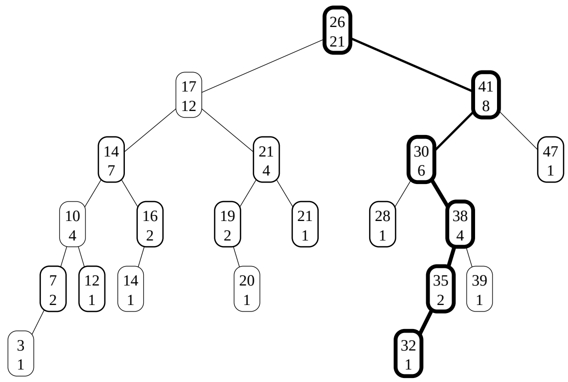
Example (RANK of element 38):



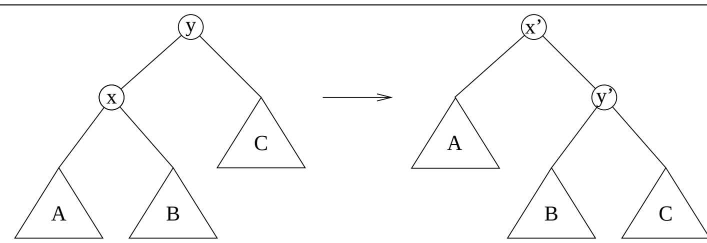
- We need to maintain the extra field during updates:
  - INSERT(i):

* Search down one root-leaf part as usual for position where i should be inserted.
* Increment $size(x)$ for all nodes x on root-leaf path (these are the *only* nodes for which the size field change).

  Example (Insertion of element 32)



* Rebalancing using Red-black tree rules—recall that we do $O(\log n)$ recolorings and $O(1)$ rotations:
  · Color change rules do not affect extra field
  · Rotations do affect size extra fields but we can still easily perform a rotation in $O(1)$ time



$size(y') = size(root(B)) + size(root(C)) + 1$
$size(x') = size(y) = size(root(A)) + size(root(B)) + size(root(C)) + 2$

$\Downarrow$
INSERT performed in $O(\log n)$ time.

– DELETE(i):
  * Find element to delete and decrement size field on one root-leaf path (recall that conceptually we always delete a node with at most one child).
  * Rebalance using rotations.

  $\Downarrow$
  DELETE performed in $O(\log n)$ time.

- Note: The key to maintaining the size field during updates is that the field of node x only depend on the field of the children of x $\Rightarrow$
  – Insertion or deletion only affect one root-leaf path.

- Rotations can be handled in $O(1)$ time locally.

- In general we can easily prove the following:

  A field f in a red-black tree can be maintained in $O(\log n)$ time during updates if $f(x)$ can be computed using only information in x, $left(x)$ and $right(x)$ (including $f(left(x))$ and $f(right(x))$)

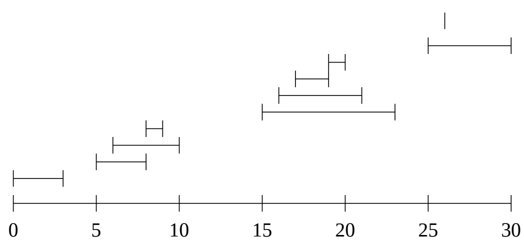
  - When changing field in a node x, f can only change for the $O(\log n)$ ancestors of x on the path to the root.
  - Rotations can be handled in $O(1)$ time locally.

3 Interval Tree

- We now consider a slightly more complicated augmentation. We want so solve the following problem:

  - Maintain a set of n intervals $[i_1, i_2]$ such that one of the intervals containing a query point q (if any) can be found efficiently.

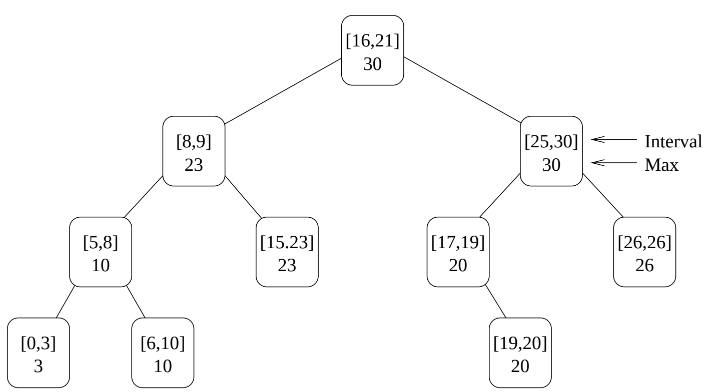
  Example: A set of intervals. A query with $q = 9$ returns $[6, 10]$ or $[8, 9]$. A query with $q = 23$ returns $[15, 23]$.



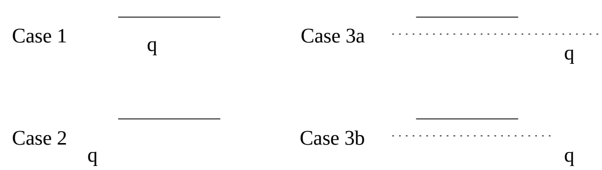
- To solve the problem we use the so-called "Interval tree":

  - Red-black tree with intervals in nodes
    * Key is left endpoint
  - Node x augmented with maximal right endpoint in subtree rooted in x

  Example: Interval tree on intervals from previous figure:



- We can maintain the interval tree dynamically during insertions and deletions in $O(\log n)$ time
  - because augmented field in x only depends on augmented fields in the children of x and the interval stored in x.
  - $max(x) = \max(rightendpoint(x), max(left(x)), max(right(x)))$
- We can also answer a query in $O(\log n)$ time:
  1. We first check if q is contained in interval stored in root r—if it is we are done.
  2. Next we check if q is on left side of left endpoint of interval in r—if it is we recursively search in left subtree (q cannot be contained in any interval in right subtree).
  3. If q is to the right of left endpoint of interval in r we have two cases:
     (a) If $max(left(r)) > q$ there must be a segment in left subtree containing q and we recurse left.
     (b) If $max(left(r)) < q$ there is no segment in left subtree containing q and we recurse right.



> QUERY(x, q)
>   IF q contained in x interval THEN Return x
>   IF $max(left(x)) \geq q$ THEN
>   Return Query$(left(x), q)$
>   ELSE
>   Return Query$(right(x), q)$
>   FI

$\Downarrow$

We search down one root-leaf path $\Rightarrow O(\log n)$ time.

Example: Query with $q = 23$: