Union-Find Algorithms

- network connectivity
- quick find
- quick union
- improvements
- applications

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.
- Define the problem.
- Find an algorithm to solve it.
- Fast enough?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method

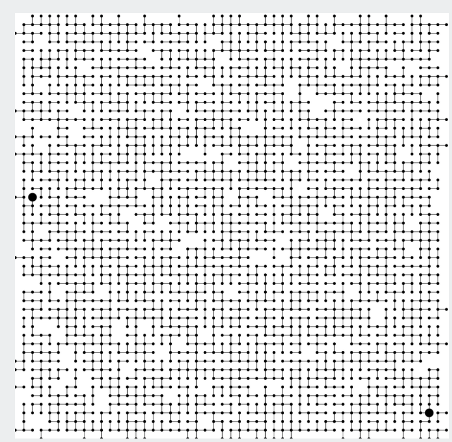
Mathematical models and computational complexity

READ Chapter One of Algs in Java

- network connectivity
- quick find
- quick union
- improvements
- applications

Network connectivity

Basic abstractions
- set of objects
- union command: connect two objects
- find query: is there a path connecting one object to another?



Objects

Union-find applications involve manipulating objects of all types.

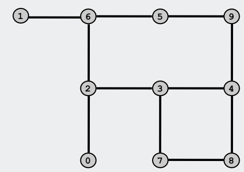
- Computers in a network.
- Web pages on the Internet.
- Transistors in a computer chip.
- Variable name aliases.
- Pixels in a digital photo.
- Metallic sites in a composite system.

stay tuned

When programming, convenient to name them 0 to N-1.

- Hide details not relevant to union-find.
- Integers allow quick access to object-related info.
- Could use symbol table to translate from object names

use as array index



Union-find abstractions

Simple model captures the essential nature of connectivity.

- Objects.

`0 1 2 3 4 5 6 7 8 9` grid points

- Disjoint sets of objects.

`0 1 { 2 3 9 } { 5 6 } 7 { 4 8 }` subsets of connected grid points

- Find query: are objects 2 and 9 in the same set?

`0 1 { 2 3 9 } { 5-6 } 7 { 4-8 }` are two grid points connected?

- Union command: merge sets containing 3 and 8.

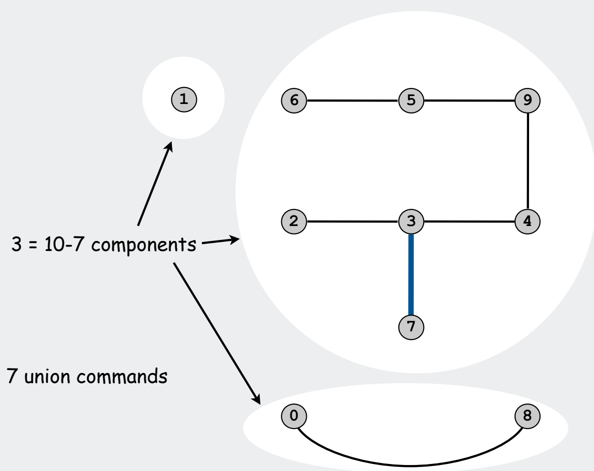
`0 1 { 2 3 4 8 9 } { 5-6 } 7` add a connection between two grid points

Connected components

Connected component: set of mutually connected vertices

Each union command reduces by 1 the number of components

in	out
3 4	3 4
4 9	4 9
8 0	8 0
2 3	2 3
5 6	5 6
2 9	
5 9	5 9
7 3	7 3

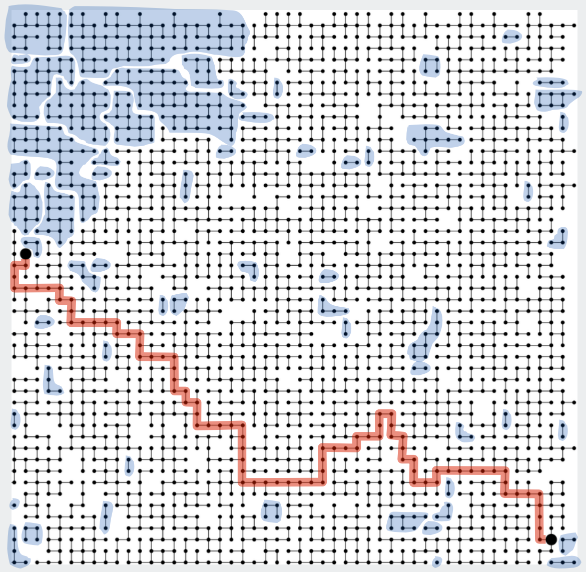


Network connectivity: larger example

Network connectivity: larger example

find(u, v) ?

true



63 components

Union-find abstractions

- Objects.
- Disjoint sets of objects.
- Find queries: are two objects in the same set?
- Union commands: replace sets containing two items by their union

Goal. Design efficient data structure for union-find.

- Find queries and union commands may be intermixed.
- Number of operations M can be huge.
- Number of objects N can be huge.

- network connectivity
- **quick find**
- quick union
- improvements
- applications

Quick-find [eager approach]

Data structure.
- Integer array `id[]` of size `N`.
- Interpretation: `p` and `q` are connected if they have the same id.

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	9	9	6	6	7	8	9

5 and 6 are connected
2, 3, 4, and 9 are connected

Quick-find [eager approach]

Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `p` and `q` are connected if they have the same id.

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	9	9	6	6	7	8	9

5 and 6 are connected
2, 3, 4, and 9 are connected

Find. Check if `p` and `q` have the same id.

id[3] = 9; id[6] = 6
3 and 6 not connected

Union. To merge components containing `p` and `q`, change all entries with `id[p]` to `id[q]`.

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	6	6	6	6	6	7	8	6

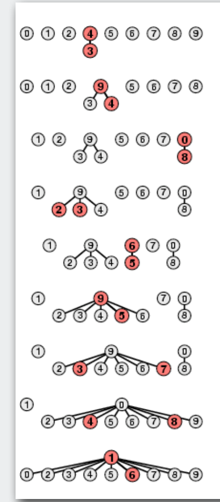
union of 3 and 6
2, 3, 4, 5, 6, and 9 are connected

problem: many values can change

Quick-find example

3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	9	9	5	6	7	8	9
8-0	0	1	2	9	9	5	6	7	0	9
2-3	0	1	9	9	9	5	6	7	0	9
5-6	0	1	9	9	9	6	6	7	0	9
5-9	0	1	9	9	9	9	9	7	0	9
7-3	0	1	9	9	9	9	9	9	0	9
4-8	0	1	0	0	0	0	0	0	0	0
6-1	1	1	1	1	1	1	1	1	1	1

problem: many values can change



Quick-find: Java implementation

```
public class QuickFind
{
   private int[] id;

   public QuickFind(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }

   public boolean find(int p, int q)
   {
      return id[p] == id[q];
   }

   public void unite(int p, int q)
   {
      int pid = id[p];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = id[q];
   }
}
```

set id of each object to itself

1 operation

N operations

Quick-find is too slow

Quick-find algorithm may take ~MN steps to process M union commands on N objects

Rough standard (for now).

- 10^9 operations per second.
- 10^9 words of main memory.
- Touch all words in approximately 1 second.

a truism (roughly) since 1950 !

Ex. Huge problem for quick-find.

- 10^{10} edges connecting 10^9 nodes.
- Quick-find takes more than 10^{19} operations.
- 300+ years of computer time!

Paradoxically, quadratic algorithms get worse with newer equipment.

- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

- network connectivity
- quick find
- **quick union**
- improvements
- applications

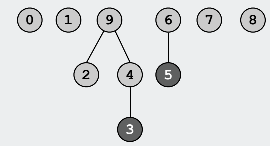
Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	4	9	6	6	7	8	9



3's root is 9; 5's root is 6

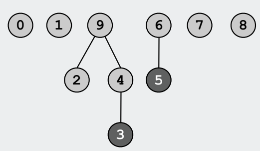
Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of size `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change

i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	4	9	6	6	7	8	9



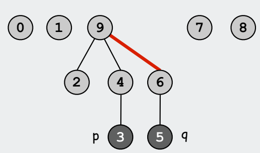
3's root is 9; 5's root is 6
3 and 5 are not connected

Find. Check if p and q have the same root.

Union. Set the id of q's root to the id of p's root.

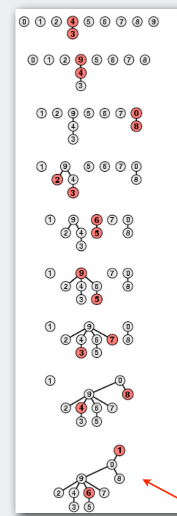
i	0	1	2	3	4	5	6	7	8	9
id[i]	0	1	9	4	9	6	9	7	8	9

only one value changes



Quick-union example

3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9
5-9	0	1	9	4	9	6	9	7	0	9
7-3	0	1	9	4	9	6	9	9	0	9
4-8	0	1	9	4	9	6	9	9	0	0
6-1	1	1	9	4	9	6	9	9	0	0



problem: trees can get tall

Quick-union: Java implementation

```
public class QuickUnion
{
   private int[] id;

   public QuickUnion(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   private int root(int i)
   {
      while (i != id[i]) i = id[i];
      return i;
   }


   public boolean find(int p, int q)
   {
      return root(p) == root(q);
   }

   public void unite(int p, int q)
   {
      int i = root(p);
      int j = root(q);
      id[i] = j;
   }
}
```

time proportional to depth of i

time proportional to depth of p and q

time proportional to depth of p and q

Quick-union is also too slow

Quick-find defect.
- Union too expensive (N steps).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.
- Trees can get tall.
- Find too expensive (could be N steps)
- Need to do find to do union

algorithm	union	find
Quick-find	N	1
Quick-union	N*	N ← worst case

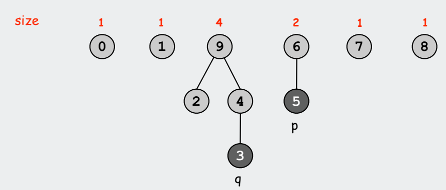
* includes cost of find

- network connectivity
- quick find
- quick union
- **improvements**
- applications

Improvement 1: Weighting

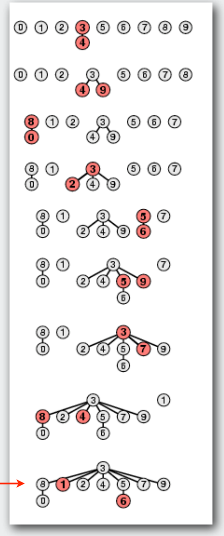
Weighted quick-union.
- Modify quick-union to avoid tall trees.
- Keep track of size of each component.
- Balance by linking small tree below large one.

Ex. Union of 5 and 3.
- Quick union: link 9 to 6.
- Weighted quick union: link 6 to 9.



Weighted quick-union example

```
3-4   0 1 2 3 3 5 6 7 8 9
4-9   0 1 2 3 3 5 6 7 8 3
8-0   8 1 2 3 3 5 6 7 8 3
2-3   8 1 3 3 3 5 6 7 8 3
5-6   8 1 3 3 3 5 5 7 8 3
5-9   8 1 3 3 3 3 5 7 8 3
7-3   8 1 3 3 3 3 5 3 8 3
4-8   8 1 3 3 3 3 5 3 3 3
6-1   8 3 3 3 3 3 5 3 3 3
```



no problem: trees stay flat

Weighted quick-union: Java implementation

Java implementation.
- Almost identical to quick-union.
- Maintain extra array `sz[]` to count number of elements in the tree rooted at i.

Find. Identical to quick-union.

Union. Modify quick-union to
- merge smaller tree into larger tree
- update the `sz[]` array.

```
if (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else sz[i] < sz[j] { id[j] = i; sz[i] += sz[j]; }
```


Weighted quick-union analysis

Analysis.
- Find: takes time proportional to depth of p and q.
- Union: takes constant time, given roots.
- Fact: depth is at most lg N. [needs proof]

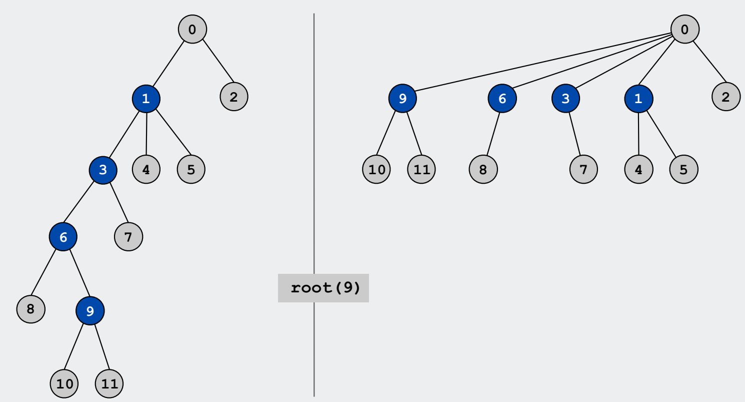
Data Structure	Union	Find
Quick-find	N	1
Quick-union	N *	N
Weighted QU	lg N *	lg N

* includes cost of find

Stop at guaranteed acceptable performance? No, easy to improve further.

Improvement 2: Path compression

Path compression. Just after computing the root of `i`, set the `id` of each examined node to `root(i)`.



Weighted quick-union with path compression

Path compression.

- Standard implementation: add second loop to `root()` to set the id of each examined node to the root.
- Simpler one-pass variant: make every other node in path point to its grandparent.

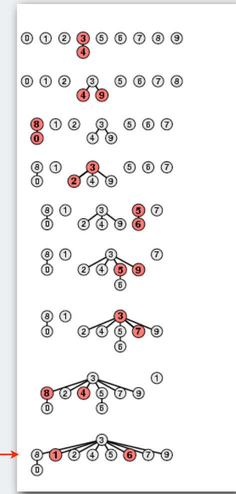
```
public int root(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

Weighted quick-union with path compression

3-4	0	1	2	3	3	5	6	7	8	9
4-9	0	1	2	3	3	5	6	7	8	3
8-0	8	1	2	3	3	5	6	7	8	3
2-3	8	1	3	3	3	5	6	7	8	3
5-6	8	1	3	3	3	5	5	7	8	3
5-9	8	1	3	3	3	3	5	7	8	3
7-3	8	1	3	3	3	3	5	3	8	3
4-8	8	1	3	3	3	3	5	3	3	3
6-1	8	3	3	3	3	3	3	3	3	3



no problem: trees stay VERY flat

WQUPC performance

Theorem. Starting from an empty data structure, any sequence of M union and find operations on N objects takes O(N + M lg* N) time.

- Proof is very difficult.
- But the algorithm is still simple!

number of times needed to take the lg of a number until reaching 1

Linear algorithm?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

because lg* N is a constant in this universe

N	lg* N
1	0
2	1
4	2
16	3
65536	4
265536	5

Amazing fact:

- In theory, no linear linking strategy exists

Summary

Algorithm	Worst-case time
Quick-find	M N
Quick-union	M N
Weighted QU	N + M log N
Path compression	N + M log N
Weighted + path	(M + N) lg* N

M union-find ops on a set of N objects

Ex. Huge practical problem.

- 10^{10} edges connecting 10^9 nodes.
- WQUPC reduces time from 3,000 years to 1 minute.
- Supercomputer won't help much.
- Good algorithm makes solution possible.

WQUPC on Java cell phone beats QF on supercomputer!

Bottom line.
WQUPC makes it possible to solve problems that could not otherwise be addressed

- network connectivity
- quick find
- quick union
- improvements
- **applications**

Union-find applications

✓ Network connectivity.
- Percolation.
- Image processing.
- Least common ancestor.
- Equivalence of finite state automata.
- Hinley-Milner polymorphic type inference.
- Kruskal's minimum spanning tree algorithm.
- Games (Go, Hex)
- Compiling equivalence statements in Fortran.

Percolation

A model for many physical systems
- N-by-N grid.
- Each square is vacant or occupied.
- Grid percolates if top and bottom are connected by vacant squares.

percolates   does not percolate

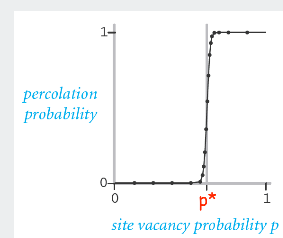
model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

Percolation phase transition

Likelihood of percolation depends on site vacancy probability p

p low: does not percolate   p high: percolates

Experiments show a threshold p*
- p > p*: almost certainly percolates
- p < p*: almost certainly does not percolate



Q. What is the value of p* ?

UF solution to find percolation threshold

- Initialize whole grid to be "not vacant"
- Implement "make site vacant" operation
  that does `union()` with adjacent sites
- Make all sites on top and bottom rows vacant
- Make random sites vacant until `find(top, bottom)`
- Vacancy percentage estimates p*

top

0	0	0	0	0	0	0	0	0	0	0	0
2	3	4	5	6	7	8	9	10	11	12	0
14	15	16	16	16	16	20	21	22	23	24	0
14	14	28	29	30	31	32	33	34	35	36	0
14	39	40	1	42	43	32	45	46	1	1	49
50	1	52	1	54	55	56	57	58	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1

bottom

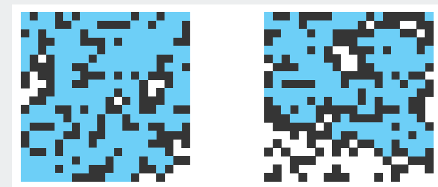
vacant

not vacant

Percolation

Q. What is percolation threshold p* ?
A. about 0.592746 for large square lattices.

percolation constant known only via simulation



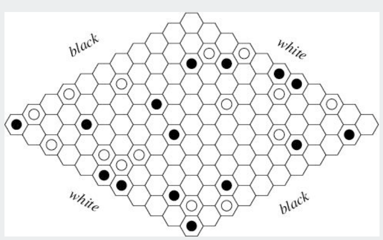
percolates   does not percolate

Q. Why is UF solution better than solution in IntroProgramming 2.4?

Hex

Hex. [Piet Hein 1942, John Nash 1948, Parker Brothers 1962]

- Two players alternate in picking a cell in a hex grid.
- Black: make a black path from upper left to lower right.
- White: make a white path from lower left to upper right.



Reference: http://mathworld.wolfram.com/GameofHex.html

Union-find application. Algorithm to detect when a player has won.

Subtext of today's lecture (and this course)

Steps to developing an usable algorithm.

- Define the problem.
- Find an algorithm to solve it.
- Fast enough?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method

Mathematical models and computational complexity

READ Chapter One of Algs in Java