# Heaps. Heapsort.

(CLRS 6)

## 1 Introduction

- We have discussed several fundamental algorithms (sorting, selection, etc)
- We will now turn to *data structures*; They play an important role in algorithms design.
  - Today we discuss priority queues and next time structures for maintaining ordered sets.

## 2 Priority Queue

- A priority queue supports the following operations on a set $S$ of $n$ elements:
  - Insert: Insert a new element $e$ in $S$
  - FindMin: Return the minimal element in $S$
  - DeleteMin: Delete the minimal element in $S$
- Sometimes we are also interested in supporting the following operations:
  - Change: Change the key (priority) of an element in $S$
  - Delete: Delete an element from $S$
- Priority queues have many applications, e.g. in discrete event simulation, graph algorithms
- We can obviously sort using a priority queue:
  - Insert all elements using Insert
  - Delete all elements in order using FindMin and DeleteMin

## 3 Priority Queue implementations

### 3.1 A Priority Queue with an Array or List

- The first implementation that comes to mind is ordered array:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 5 | 6 | 7 | 8 | 9 | 11 | 12 | 15 | 17 |

  - FindMin can be performed in $O(1)$ time

- $-$ DeleteMin and Insert takes $O(n)$ time since we need to expand/compress the array after inserting or deleting element.

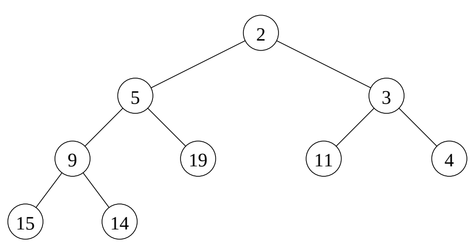
- If the array is unordered all operations take $O(n)$ time.
- We could use double linked sorted list instead of array to avoid the $O(n)$ expansion/compression cost
  - $-$ but Insert can still take $O(n)$ time.

## 3.2 A Priority Queue with a Heap

- The common way of implementing a priority queue is using a heap
- Heap definition:

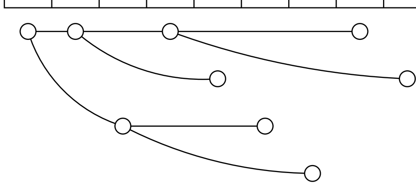
  - $-$ Perfectly balanced binary tree
    - $*$ lowest level can be incomplete (but filled from left-to-right)
  - $-$ For all nodes $v$ we have key($v$)$\geq$key(parent($v$))

- Example:



- Heap can be implemented (stored) in two ways (at least)
  - $-$ Using pointers
  - $-$ In an array level-by-level, left-to-right
    Example:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 5 | 3 | 9 | 19 | 11 | 4 | 15 | 14 |



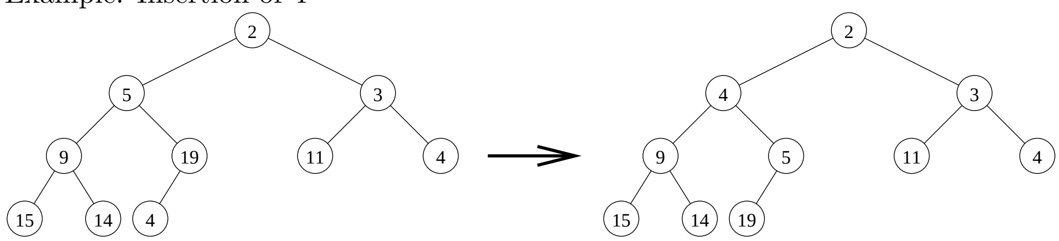
    - $*$ the left and right children of node in entry $i$ are in entry $2i$ and $2i+1$, respectively
    - $*$ the parent of node in entry $i$ is in entry $\lfloor \frac{i}{2} \rfloor$

- Properties of heap:
  - $-$ Height $\Theta(\log n)$

– Minimum of $S$ is stored in root

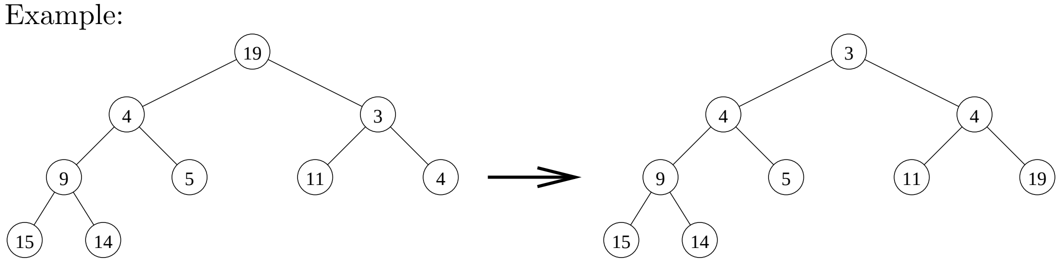
- Operations:
  - Insert
    - Insert element in new leaf in leftmost possible position on lowest level
    - Repeatedly swap element with element in parent node until heap order is reestablished (up-heapify)

      Example: Insertion of 4




  - FindMin
    - Return root element
  - DeleteMin
    - Delete element in root
    - Move element from rightmost leaf on lowest level to the root (and delete leaf)
    - Repeatedly swap element with the smaller of the children elements until heap order is reestablished (down-heapify)

      Example:




  - Change and Delete can be handled similarly in $O(\log n)$ time
    - Note: Assuming that we know the element to be changed/deleted (we cannot search in a heap!!)

- **Correctness:** Exercise.
- **Running time:** All operations traverse at most one root-leaf path $\Rightarrow O(\log n)$ time.
- Sorting using heap (*HeapSort*) takes $\Theta(n \log n)$ time.
  - $n \cdot O(\log n)$ time to insert all elements (build the heap)
  - $n \cdot O(\log n)$ time to output sorted elements
- Sometimes we would like to build a heap faster than $O(n \log n)$

- BUILDHEAP
  - Insert elements in any order in perfectly balanced tree
  - DOWN-HEAPIFY all nodes level-by-level, bottom-up
- Correctness:
  - Induction on height of tree: When doing level $i$, all trees rooted at level $i-1$ are heaps.
- Analysis:
  - The leaves are at height 0, the root is at height $\log n$
  - $n$ elements $\Rightarrow \leq \lceil \frac{n}{2} \rceil$ leaves $\Rightarrow \lceil \frac{n}{2^h} \rceil$ elements at height $h$
  - Cost of DOWN-HEAPIFY on a node at height $h$ is $h$
  - Total cost: $\sum_{i=1}^{\log n} h \cdot \lceil \frac{n}{2^h} \rceil = \Theta(n) \cdot \sum_{i=1}^{\log n} \frac{h}{2^h}$
  - It can be shown that $\sum_{i=1}^{\log n} \frac{h}{2^h} = O(1) \Longrightarrow$ the total buildheap cost is $\Theta(n)$

  - Computing $\sum_{i=1}^{n} \frac{h}{2^h}$ and $\sum_{i=1}^{\infty} \frac{h}{2^h}$
    - Differentiate $\sum_{h=0}^{n} x^h = \frac{1-x^{n+1}}{1-x}$, respectively $\sum_{h=0}^{\infty} x^h = \frac{1}{1-x}$ (assuming $|x| < 1$)
    - $\sum_{h=0}^{\infty} hx^{h-1} = \frac{1}{(x-1)^2} \Rightarrow \sum_{h=0}^{n} hx^h = \frac{x}{(x-1)^2} \Rightarrow \sum_{h=0}^{n} \frac{h}{2^h} = \frac{1/2}{(1/2-1)^2} = O(1)$