Greedy Algorithms

CLRS 16.1-16.2

1 Greedy Algorithms

- We have previously discussed how to speed up optimization problems using the technique of *dynamic programming*:
  - The problem must have the *optimal substructure property*: the optimal solution to the problem contains within it optimal solutions to smaller subproblems.
  - Typically the number of different subproblems is polynomial, but the recursive algorithm implementing the recursive formulation of the problem is exponential. This is because of overlapping calls to same subproblem.
  - Idea: If same subproblem is solved several times, use table to store result of a subproblem the first time it is computed and never compute it again.
  - Alternatively, we can think about filling up a table of subproblem solutions from the bottom.
- Another, simpler and often less powerful (and less well-defined), technique that uses the same feature of optimal substructure is *greediness*.
- Like in the case of dynamic programming, we will introduce greedy algorithms via an example.

1.1 Activity Selection

- Problem: Given a set $A = \{A_1, A_2, \cdots, A_n\}$ of n activities with start and finish times (s_i, f_i), $1 \leq i \leq n$, select maximal set S of "non-overlapping" activities.
  - One can think of the problem as corresponding to scheduling the maximal number of classes (given their start and finish times) in one classroom.
- Dynamic programming solution: $O(n^3)$ read the textbook.
- Greedy solution:
  - Sort activity by finish time (let $A_1, A_2, \cdots, A_n$ denote sorted sequence)
  - Pick first activity A_1
  - Remove all activities with start time before finish time of A_1
  - Recursively solve problem on remaining activities.

- Program:

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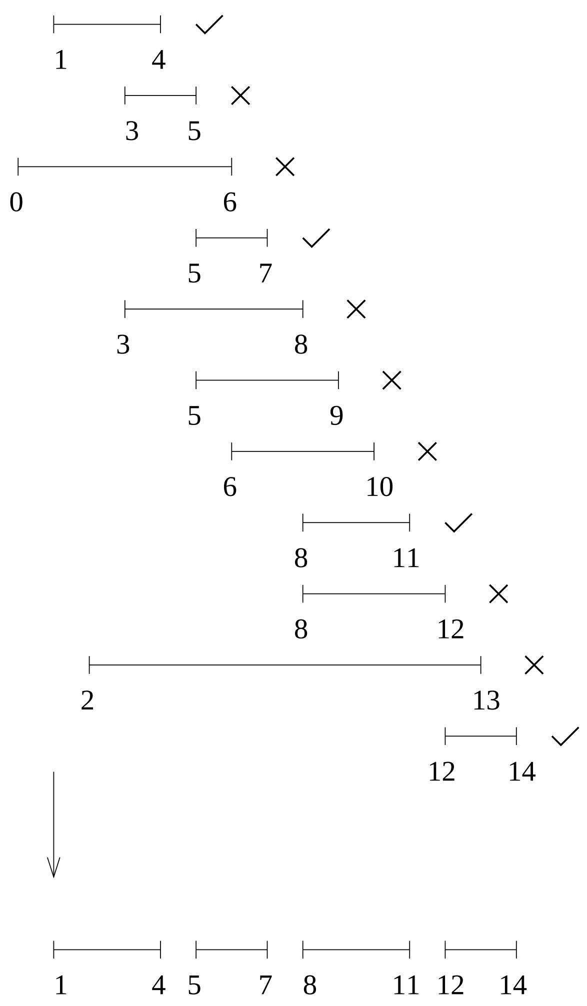
Sort $A$ by finish time
$S = \{A_1\}$
$j = 1$
FOR $i = 2$ to $n$ DO
    IF $s_i \geq s_j$ THEN
        $S = S \cup \{A_i\}$
        $j = i$
    FI
OD
```

- Example:

  - 11 activities sorted by finish time: $(1,4), (3,5), (0,6), (5,7), (3,8), (5,9),$ $(6,10), (8,11), (8,12), (2,13), (12,14)$



- Running time is obviously $O(n \log n)$.
- Is algorithm correct?
  - Output is set of non-overlapping activities, but is it the largest possible?
- Proof of correctness:
  - Given activities $A = \{A_1, A_2, \cdots, A_n\}$ ordered by finish time, there is an optimal solution containing A_1:
    * Suppose $S \subseteq A$ is optimal solution
    * If $A_1 \in S$, we are done
    * If $A_1 \notin S$:
      · Let first activity in S be A_k
      · Make new solution $S \setminus \{A_k\} \cup \{A_1\}$ by removing A_k and using A_1 instead
      · This is valid solution ($f_1 < f_k$) of maximal size ($|S \setminus \{A_k\} \cup \{A_1\}| = |S|$)
  - S is an optimal solution for A containing $A_1 \Rightarrow S' = S \setminus \{A_1\}$ optimal solution for $A' = \{A_i \in A : s_j \geq f_1\}$ (e.g. after choosing A_1 the problem reduces to finding optimal solution for activities not overlapping with A_1)

    - Suppose we have solution S'' to A' such that $|S''| > |S'| = |S| - 1$
    - $S''' = S'' \cup \{A_1\}$ would be solution to A
    - Contradiction since we would have $|S'''| > |S|$
  - Correctness follows by induction on size of S

2 Comparison of greedy technique with dynamic programming

- Both techniques use optimal substructure (optimal solution "contains optimal solution for subproblems within it").
- In dynamic programming, solution depends on solution to subproblems.That is, compute the optimal solutions for each possible choice and thencompute the optimal way to combine things together.
- In greedy algorithm we choose what looks like best solution at any given moment and recurse (choice does not depend on solution to subproblems).

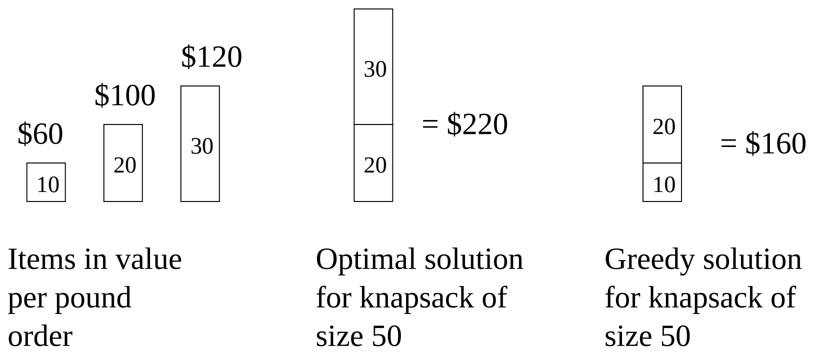
  Note: Shortsightedness: Always go for seemingly next best thing, optimizing the present without regard for the future, and never change past choices.
- Any problem that can be solved by a greedy algorithm can be solved by dynamic programming, but not the other way around.
- It is often hard to figure out when being greedy works!
- How do we know if being greedy works? Try dynamic programming first and understand the choices. Try to find out if there is a locally best choice, i.e. a choice that looks better than the others (without computing recursive solutions to subproblems). Now try to prove that it works correctly.
- Greedy correctness proof: It is enough to prove that there exists an optimal solution which contains the greedy choice. That is, prove that, having made the greedy choice, what remains is a subproblem with the property that if we combine the optimal solution to the subproblem with the greedy choice, we get an optimal solution for the original problem. Typically this is proved by contradiction.

Example:

- $0-1$ KNAPSACK PROBLEM: Given n items, with item i being worth \$ v_i and having weight w_i pounds, fill knapsack of capacity w pounds with maximal value.
- FRACTIONAL KNAPSACK PROBLEM: As $0-1$ KNAPSACK PROBLEM but we can take fractions of items.
- Problems appear very similar, but only FRACTIONAL KNAPSACK PROBLEM can be solved greedily:
  - Compute value per pound $\frac{v_i}{w_i}$ for each item
  - Sort items by value per pound.

  - Fill knapsack greedily (take objects in order)
    $\Downarrow$
    $O(n \log n)$ time, easy to show that solution is optimal.

- Example that $0-1$ KNAPSACK PROBLEM cannot be solved greedily:


$60
$100
$120
10
20
30
Items in value per pound order
30
20
= $220
Optimal solution for knapsack of size 50
20
10
= $160
Greedy solution for knapsack of size 50

  Note: In FRACTIONAL KNAPSACK PROBLEM we can take $\frac{2}{3}$ of \$120 object and get \$240 solution.

- $0-1$ KNAPSACK PROBLEM can be solved in time $O(n \cdot w)$ using dynamic-programming (homework).