Hashing

(CLRS 11.1-11.3)

1 Maintaining ordered set

- Last time we started discussing the problem of maintaining an ordered set S under operations
  - Search
  - Insert
  - Delete
  - Successor
  - Predecessor
- We discussed several implementations
  - Array
  - Linked list
  - Skip lists
- We saw that in skip list all operations have *expected* running time $O(\log n)$
  - Next time we will discuss a data structure (red-black tree) with *worst-case* $O(\log n)$ running time.
- We can argue that $\Theta(\log n)$ time is optimal for searching in the decision tree model

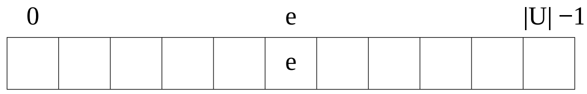
  Recall decision tree model:

  > - Binary tree where each node is labeled $a_i \leq a_j$
  > - Execution corresponds to root-leaf path
  > - Leaf contains result of computation

  - Decision trees correspond to algorithms where we are only allowed to use comparison to gain knowledge about input.
  - Decision tree for search must have n leaves (one for each element)
    $\Downarrow$
    Tree must have height $\Omega(\log n)$
- In the case of sorting, we saw that we could beat the $\Omega(n \log n)$ decision tree lower bound using *Indirect Addressing* (Radix sort)
  - we can also use indirect addressing idea on ordered set problem.

2 Direct Addressing

- Store element e in cell e of array (we assume elements are integers)

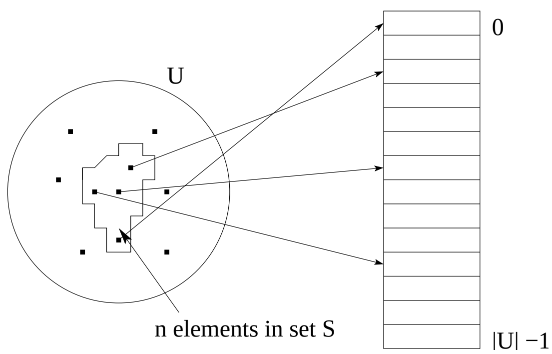


  - Insert/Delete/Search in O(1) time
  - Predecessor/Successor in $O(|U|)$ time ($|U|$ is the size of "universe" U)

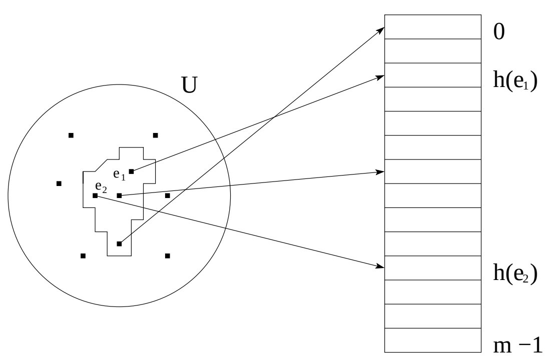
- Note: We could make predecessor/successor efficient by linking neighbor elements, but then $Insert/Delete$ becomes $O(|U|)$
- Problem is that $|U|$ can be huge and often $|U| >> n$
  - 32 bit integers $\Rightarrow |U| = 2^{32}$
- We can reduce space use using "hashing"

3 Hashing

- To introduce hashing, we look at direct addressing in a slightly different way :

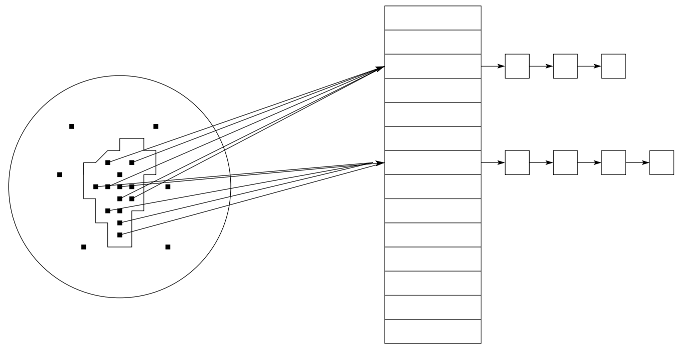


- The main idea is to fix the table size to $m = O(n)$)
  - now element e cannot be stored in cell e
    $\Downarrow$
    We introduce *hash function* $h(e) : U \rightarrow \{0, 1, ...m-1\}$



We call the array the *hash table*

- Problem is of course that several elements can be stored in same cell ($m < |U|$)
  - We call such an event a *collision*
- We solve this problem using *chaining*
  - Elements mapping to same cell are stored in linked list



- worst-case: INSERT in $O(1)$, DELETE/SEARCH in $O(\text{max chain length})$
- PREDECESSOR/SUCCESSOR in $O(m+n)$ since we have to look in all cells and chains (Note : We assume we can compute $h(e)$ in $O(1)$ time)
- Note: PREDECESSOR/SUCCESSOR bounds are very bad (we will not discuss them further in the following)
  - We call a data structure only supporting INSERT/DELETE/SEARCH a *Dictionary*
  - In a dictionary, order does not really matter
  - Lots of applications of dictionaries, e.g.
    - Symbol table in compilers
    - IP addresses to machine-name table
- Performance of hashing depends on how well $h(e)$ spreads the elements in the hash table
  - Lets make the *simple uniform hashing* assumption

Any given element is equally likely to hash into any of the m cells

  $\Downarrow$
  - On average $\frac{n}{m}$ elements in each chain and searching takes $O(\frac{n}{m})$ on the average

  $\Downarrow$
  - If we choose $m = O(n)$ we get $O(1)$ bounds (and $O(n)$ space instead of $O(|U|)$)
- How do we choose a good hashing function?
  - Often $h(e) = e \bmod m$ is used ($e \bmod m$ is remainder of e divided by m)
    Example : $m = 12, e = 100 \Rightarrow h(e) = 4$ since $100 = 8 \cdot 12 + 4$
  - m is often chosen to be a prime number far away from a power of 2

    If $m = 2^p$ then $h(e) =$ lowest p bits in e which means that the hashing value only depends on some of the bits in e. If data is not random—not all p-bit patterns equally likely—then this might be a very bad choice, we would rather have $h(e)$ depend on all the bits

4 Universal Hashing

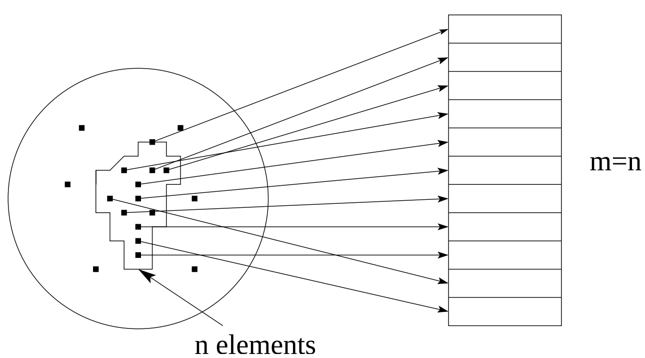
- Given hash function h, we can always find sets of elements that make hashing perform badly (n elements that map to same location)
- Like in Quick-sort and skip lists we can make sure our data structure does not perform badly on a particular input (set of inputs) using randomization
  - We choose a hash function randomly (independent of elements) from a carefully defined set of functions
    $\Downarrow$
  - no worst case inputs
  - good average case behavior
- We want the set of hash functions to be *universal*

> Let H be a finite collection of functions $U \rightarrow 0, 1, ...m - 1$.
> H is called **universal** *if and only if* for each $x, y \in U$ the number of functions $h \in H$ for which $h(x) = h(y)$ is precisely $|H|/m$.

  - If we choose h randomly from H then the probability of collision between x and y is $\frac{|H|/m}{|H|} = \frac{1}{m}$
    $\Downarrow$
  - If $m > n$, then then expected number of collisions involving element e is < 1 $\Downarrow$ INSERT/DELETE/SEARCH in $O(1)$ expected
  - Note: The book proves the above more formally and talks about how to find universal class of hash functions (not hard but requires some number theory, so we skip it)

5 Dynamic perfect hashing

- It turns out that one can even do searches in $O(1)$ *worst-case* time. Out of scope of this class.
- Idea: If set of n keys is static, we could potentially find a *perfect* hash function h



- We need to be able to store description of h compactly and compute h fast.
- Lots of research has been done on finding perfect hash functions for a given set of elements, resulting in $O(1)$ worst-case SEARCH
- The perfect hashing idea can even be made dynamic such that one also gets $O(1)$ INSERT/DELETE expected running time. Lots of recent results even improve on this.