Heaps. Heapsort.

(CLRS 6)

1 Introduction

- We have discussed several fundamental algorithms (e.g. sorting)
- We will now turn to *data structures*; Play an important role in algorithms design.
  - Today we discuss priority queues and next time structures for maintaining ordered sets.

2 Priority Queue

- A priority queue supports the following operations on a set S of n elements:
  - Insert: Insert a new element e in S
  - FindMin: Return the minimal element in S
  - DeleteMin: Delete the minimal element in S
- Sometimes we are also interested in supporting the following operations:
  - Change: Change the key (priority) of an element in S
  - Delete: Delete an element from S
- We can obviously sort using a priority queue:
  - Insert all elements using Insert
  - Delete all elements in order using FindMin and DeleteMin
- Priority queues have many applications, e.g. in discrete event simulation, graph algorithms

2.1 Array or List implementations

- The first implementation that comes to mind is ordered array:

1	3	5	6	7	8	9	11	12	15	17

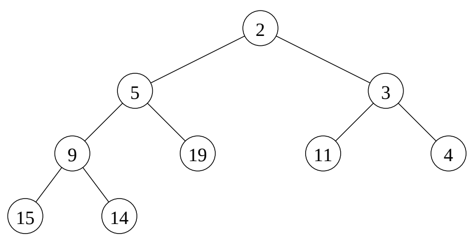
  - FindMin can be performed in $O(1)$ time
  - DeleteMin and Insert takes $O(n)$ time since we need to expand/compress the array after inserting or deleting element.
- If the array is unordered all operations take $O(n)$ time.
- We could use double linked sorted list instead of array to avoid the $O(n)$ expansion/compression cost
  - but Insert can still take $O(n)$ time.

2.2 Heap implementation

- One way of implementing a priority queue is using a heap
- Heap definition:

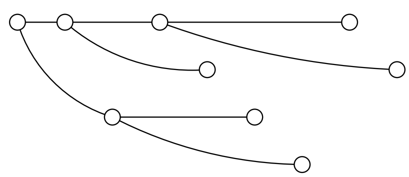
  - Perfectly balanced binary tree
    - lowest level can be incomplete (but filled from left-to-right)
  - For all nodes v we have key(v)$\geq$key(parent(v))

- Example:



- Heap can be implemented (stored) in two ways (at least)
  - Using pointers
  - In an array level-by-level, left-to-right
    Example:

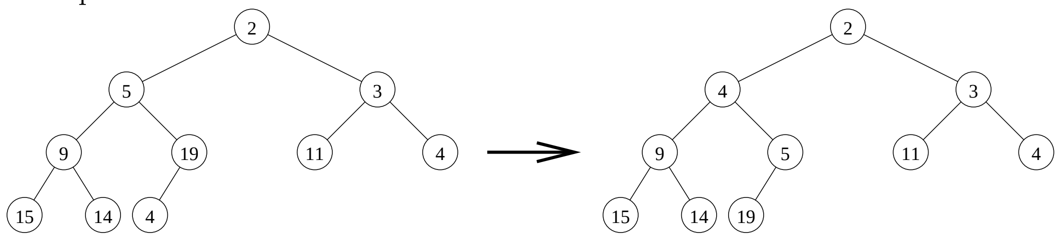
2	5	3	9	19	11	4	15	14



    - the left and right children of node in entry i are in entry $2i$ and $2i+1$, respectively
    - the parent of node in entry i is in entry $\lfloor \frac{i}{2} \rfloor$

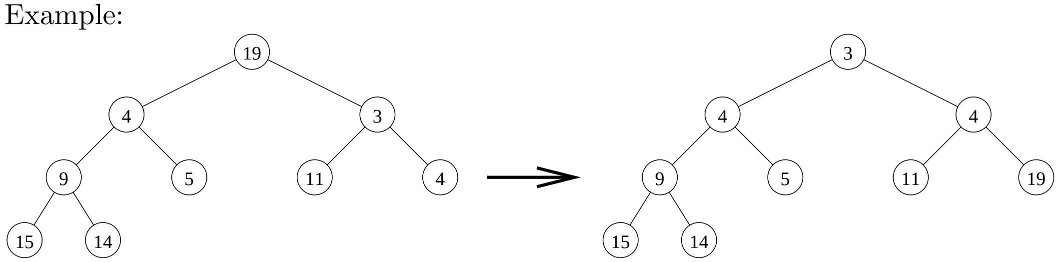
- Properties of heap:
  - Height $\Theta(\log n)$
  - Minimum of S is stored in root
- Operations:
  - Insert
    - Insert element in new leaf in leftmost possible position on lowest level
    - Repeatedly swap element with element in parent node until heap order is reestablished (up-heapify)

Example: Insertion of 4



- FindMin
    * Return root element
- DeleteMin
    * Delete element in root
    * Move element from rightmost leaf on lowest level to the root (and delete leaf)
    * Repeatedly swap element with the smaller of the children elements until heap order is reestablished (down-heapify)

    Example:



- Change and Delete can be handled similarly in $O(\log n)$ time
    * Note: Assuming that we know the element to be changed/deleted (we cannot search in a heap!!)

• **Correctness:** Exercise.

• **Running time:** All operations traverse at most one root-leaf path $\Rightarrow O(\log n)$ time.

• Sorting using heap (*HeapSort*) takes $\Theta(n \log n)$ time.

  - $n \cdot O(\log n)$ time to insert all elements (build the heap)
  - $n \cdot O(\log n)$ time to output sorted elements

• Sometimes we would like to build a heap faster than $O(n \log n)$

  - BUILDHEAP
    * Insert elements in any order in perfectly balanced tree
    * down-heapify all nodes level-by-level, bottom-up
  - Correctness:
    * Induction on height of tree: When doing level i, all trees rooted at level $i-1$ are heaps.
  - Analysis:
    * The leaves are at height 0, the root is at height $\log n$
    * n elements $\Rightarrow \leq \lceil \frac{n}{2} \rceil$ leaves $\Rightarrow \lceil \frac{n}{2^h} \rceil$ elements at height h
    * Cost of down-heapify on a node at height h is h
    * Total cost: $\sum_{i=1}^{\log n} h \cdot \lceil \frac{n}{2^h} \rceil = \Theta(n) \cdot \sum_{i=1}^{\log n} \frac{h}{2^h}$

* It can be shown that $\sum_{i=1}^{\log n} \frac{h}{2^h} = O(1) \Longrightarrow$ the total buildheap cost is $\Theta(n)$

* Computing $\sum_{i=1}^{n} \frac{h}{2^h}$ and $\sum_{i=1}^{\infty} \frac{h}{2^h}$
  · Differentiate $\sum_{h=0}^{n} x^h = \frac{1-x^{n+1}}{1-x}$, respectively $\sum_{h=0}^{\infty} x^h = \frac{1}{1-x}$ (assuming $|x| < 1$)
  · $\sum_{h=0}^{\infty} hx^{h-1} = \frac{1}{(x-1)^2} \Rightarrow \sum_{h=0}^{n} hx^h = \frac{x}{(x-1)^2} \Rightarrow \sum_{h=0}^{n} \frac{h}{2^h} = \frac{1/2}{(1/2-1)^2} = O(1)$