Binary Search Trees and Skip Lists.

(CLRS 10, 12.1-12.3)

1 Maintaining ordered set dynamically

- We want to maintain an ordered set S under operations
  - SEARCH(e): Return (pointer to) element e in S (if $e \in S$)
  - INSERT(e): Insert element e in S
  - DELETE(e): Delete element e from S
  - SUCCESSOR(e): Return (pointer to) minimal element in S larger than e
  - PREDECESSOR(e): Return (pointer to) maximal element in S smaller than e

1.1 Ordered array implementation

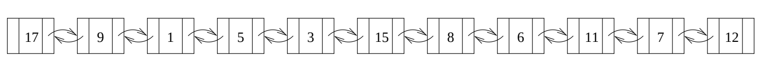
- The first implementation that comes to mind is the ordered array:

1	3	5	6	7	8	9	11	12	15	17

  - SEARCH can be performed in $O(n)$ time by scanning through array or in $O(\log n)$ time using binary search
  - PREDECESSOR/SUCCESSOR can be performed in $O(\log n)$ time like searching
  - INSERT/DELETE takes $O(n)$ time since we need to expand/compress the array after finding the position of e

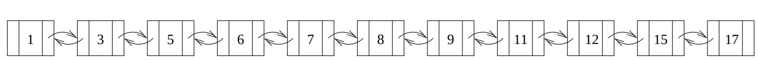
1.2 Double linked list implementation

- Unordered list



  - SEARCH takes $O(n)$ time since we have to scan the list
  - PREDECESSOR/SUCCESSOR takes $O(n)$ time
  - INSERT takes $O(1)$ time since we can just insert e at beginning of list
  - DELETE takes $O(n)$ time since we have to perform a search before spending $O(1)$ time on deletion

- Ordered list

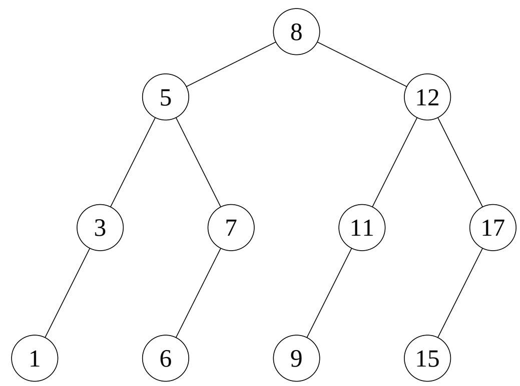


  - SEARCH takes $O(n)$ time since we cannot perform binary search

- Predecessor/Successor takes $O(n)$ time
- Insert/Delete takes $O(n)$ time since we have to perform a search to locate the position of insertion/deletion

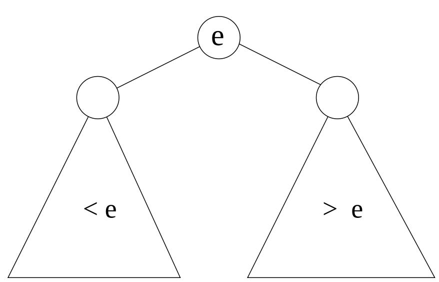
1.3 Binary search tree implementation

- Binary search naturally leads to definition of binary search tree



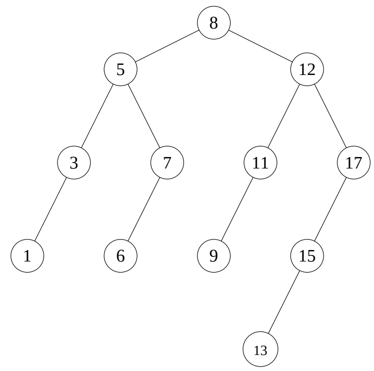
- Formal definition of search tree:

  - Binary tree with elements in nodes
  - If node v holds element e then
    - All elements in left subtree $< e$
    - All elements in right subtree $> e$



  - *Search*(e) in $O(height)$: Compare with e and recursively search in left or right subtree
  - *Insert*(e) in $O(height)$: Search for e and insert at place where search path terminates (Note: height may increase)

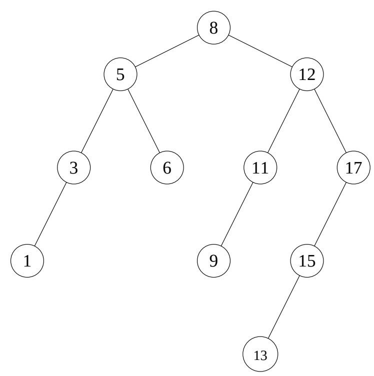
    Example: Insertion of 13



– *Delete*(e) in $O(height)$: Search for node v containing e,

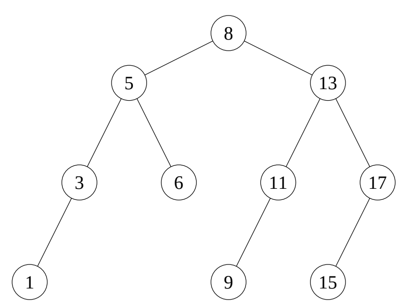
1. v is a leaf: Delete v
2. v is internal node with one child: Delete v and attach $child(v)$ to $parent(v)$

   Example: Delete 7



3. v is internal node with two children:
   * exchange e in v with successor e' in node v' (minimal element in right subtree, found by following left branches as long as possible in right subtree)
   * v' node can be deleted by case 1 or 2

   Example: Delete 12

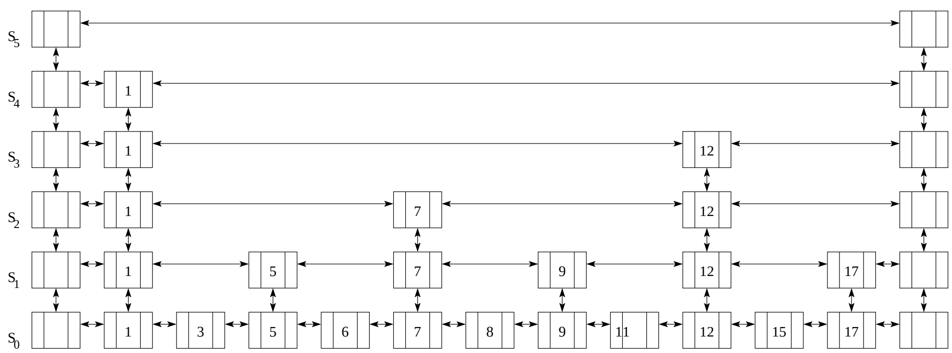


- Note:
  – Running time of all operations depend on height of tree.
  – Intuitively the tree will be nicely balanced if we do insertion and deletion randomly.
  – In worst case the height can be O(n).

2 Skip lists

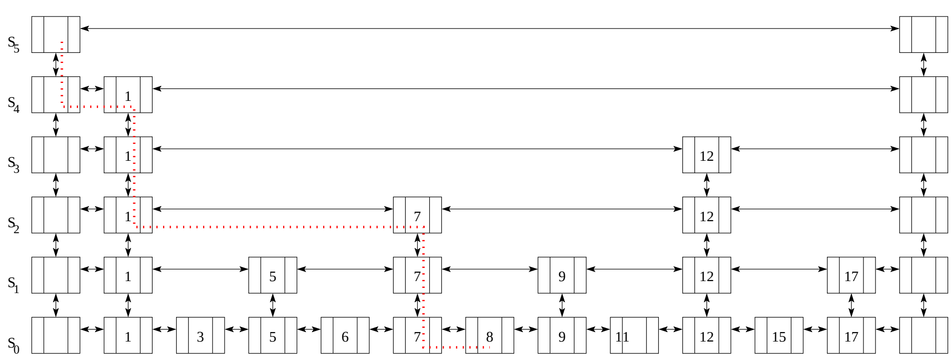
- There are several schemes for keeping search trees reasonably balanced and obtain $O(\log n)$ bounds
  - Often quite complicated—We will discuss one way (red-black trees) later.
- When we discussed Quick-sort we saw how randomization can lead to good expected running times.
  - We will now discuss how randomization can be used to obtain a very simple search structure with expected case performance $O(\log n)$ (independent of data/operations!)
- Idea in a skip list is best illustrated if we try to build a "search tree" on top of double linked list:
  - Insert elements $-\infty$ and ∞
  - Repeatedly construct double linked list (level S_i) on top of current list (level S_{i-1}) by choosing every second element (and link equal elements together)

    $\Downarrow$
  - Number of levels is $O(\log n)$



  - *Search*(e): Start at topmost left element. Repeatedly drop down one level and search forward until max element $\leq e$ is found.

    Example: Search for 8



    $O(\log n)$ time since we move at most one step to the right at each level.
  - *Predecessor*/*Successor* also in $O(\log n)$ time

- *Insert/Delete* seems hard to do in better than $O(n)$ time since we might need to rebuild the entire structure after one of the operations.

- Idea in skip list is to let level S_i consist of a randomly generated subset of elements at level S_{i-1}.
  - To decide if an element on level S_{i-1} should be on level S_i, we flip a coin and include the element if it is head.
    $\Downarrow$
    Expected size of S_1 is $\frac{n}{2}$
    Expected size of S_2 is $\frac{n}{4}$
    $\vdots$
    Expected size of S_i is $\frac{n}{2^i}$
    $\Downarrow$
    Expected height is $O(\log n)$

- Operations:
  - $Search(e)$ as before.
  - $Delete(e)$: Search to find e and delete all occurrences of e.
  - $Insert(e)$:
    * search to find position of e in S_0
    * Insert e in S_0.
    * Repeatedly flip a coin; insert e and continue to next level if it comes up head.

- Running time of all the operations is bounded by search running time
  - Down search takes $O(height) = O(\log n)$ expected.
  - Right search/scan:
    * If we scan an element on level i it cannot be on level $i+1$ (because then we would have scanned it there)
      $\Downarrow$
    * Expected number of elements we scan on level i is the expected number of times we have to flip a coin to get head
      $\Downarrow$
    * We expect to scan 2 elements on level i
      $\Downarrow$
    * Running time is $O(height) = O(\log n)$ expected.

- Note:
  - We only really need forward and down pointers.
  - Expected space use is $\sum_{i=0}^{\log n} \frac{n}{2^i} \leq n \cdot \sum_{i=0}^{\infty} \frac{1}{2^i} = O(n)$.