Lecture 2: Divide-and-Conquer and Growth of Functions

(CLRS 2.3,3)

1 Review

- Last time we discussed running time of algorithms and introduced RAM model of computation.
  - Best-case running time: the shortest running time for any input of size n
  - Worst-case running time: the longest running time for any input of size n
  - Average-case running time
- We discussed insertion sort.
- We discussed proof of correctness of insertion sort using loop invariant.
- We analyzed the running time of insertion sort in the RAM model.
  - Best-case: $k_1 n - k_2$.
  - Worst-case (and average case): $k_3 n^2 + k_4 n - k_5$
- We discussed how we are normally only interested in growth of running time:
  - Best-case: *linear* in n ($\sim n$)
  - worst-case: *quadratic* in n ($\sim n^2$).

Exercise 1 *(2.2.4-CLRS) How can you modify almost any algorithm to have a good best-case running time?*

Solution 1

2 Today

- Define formally the bf rate of growth of a function
- Introduce the divide-and-conquer algorithm design technique.
- Discuss another sorting algorithm obtained using divide-and-conquer

3 Divide-and-Conquer and Mergesort

- Can we design better than n^2 (quadratic) sorting algorithm?
- We will do so using one of the most powerful algorithm design techniques.

3.1 Divide-and-conquer

Divide-and-Conquer (Input: Problem P)

To Solve P:

1. *Divide* P into smaller problems $P_1, P_2, P_3 .....P_k$.
2. *Conquer* by solving the (smaller) subproblems recursively.
3. *Combine* solutions to $P_1, P_2, ...P_k$ into solution for P.

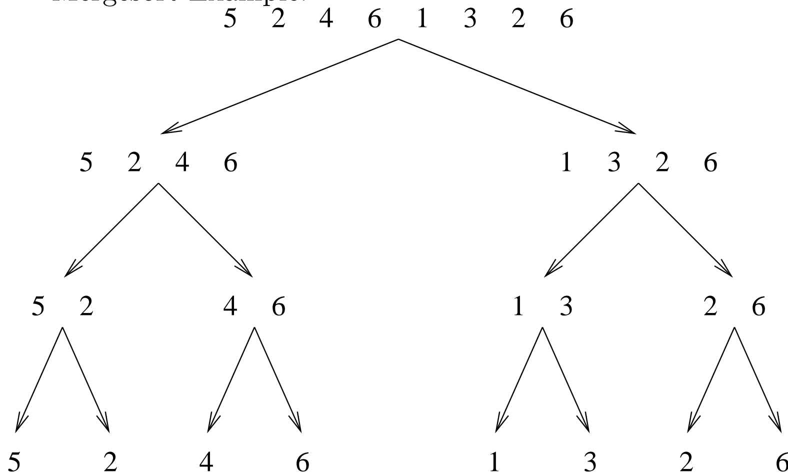
3.2 MergeSort

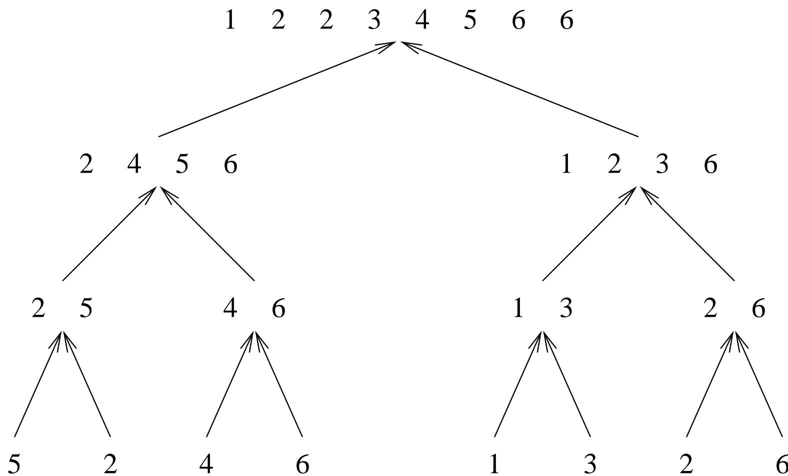
- Using divide-and-conquer, we can obtain a mergesort algorithm.
  - Divide: Divide n elements into two subsequences of $n/2$ elements each.
  - Conquer: Sort the two subsequences recursively.
  - Combine: Merge the two sorted subsequences.
- Assume we have procedure Merge(A, p, q, r) which merges sorted A[p..q] with sorted A[q+1....r]
- We can sort A[p...r] as follows (initially p=1 and r=n):

```
Merge Sort(A,p,r)
    If p < r then
        q = ⌊(p + r)/2⌋
        MergeSort(A,p,q)
        MergeSort(A,q+1,r)
        Merge(A,p,q,r)
```

- How does Merge(A, p, q, r) work?
  - Imagine merging two sorted piles of cards. The basic idea is to choose the smallest of the two top cards and put it into the output pile.
  - Running time: $(r - p)$
  - Implementation is a bit messier..

Mergesort Example:





3.3 Mergesort Correctness

- Prove that Merge() is correct (what is the invariant?)
- Assuming that Merge is correct, prove that Mergesort() is correct.
  - Induction on n

3.4 Mergesort Analysis

- To simplify things, let us assume that n is a power of 2, i.e $n = 2^k$ for some k.
- Running time of a recursive algorithm can be analyzed using a **recurrence equation/relation**. Each "divide" step yields two sub-problems of size $n/2$.

$$\begin{aligned} T(n) &\leq c_1 + T(n/2) + T(n/2) + c_2 n \\ &\leq 2T(n/2) + (c_1 + c_2 n) \end{aligned}$$

- Next class we will prove that $T(n) \leq cn \log_2 n$. Intuitively, we can see why the recurrence has solution $n \log_2 n$ by looking at the **recursion tree**: the total number of levels in the recursion tree is $\log_2 n + 1$ and each level costs linear time.

- Note: If $n \neq 2^k$ the recurrence gets more complicated, but the solution is the same. (We will often assume $n = 2^k$ to avoid complicated cases).

3.5 Algorithms matter!

Sort 10 million integers on

- 1 GHZ computer (1000 million instructions per second) using $2n^2$ algorithm.
  - $\frac{2 \cdot (10^7)^2 \ \textit{inst.}}{10^9 \ \textit{inst. per second}} = 200000$ seconds ≈ 55 hours.
- 100 MHz computer (100 million instructions per second) using $50n \log n$ algorithm.
  - $\frac{50 \cdot 10^7 \cdot \log 10^7 \ \textit{inst.}}{10^8 \ \textit{inst. per second}} < \frac{50 \cdot 10^7 \cdot 7 \cdot 3}{10^8} = 5 \cdot 7 \cdot 3 = 105$ seconds.

4 Asymptotic Growth

When we discussed Insertion Sort, we did a precise analysis of the running time and found that the worst-case is $k_3 n^2 + k_4 n - k_5$. The effort to compute all terms and the constants in front of the terms is not really worth it, because for large input the running time is dominated by the term n^2. Another good reason for not caring about constants and lower order terms is that the RAM model is not completely realistic anyway (not all operations cost the same).

$$k_3 n^2 + k_4 n - k_5 \sim n^2$$

Basically, we look at the running time of an algorithm when the input size n is large enough so that constants and lower-order terms do not matter. This is called **aymptotic analysis of algorithms**.

Now we would like to formalize this idea (It is easy to see that $n+2 \sim n$, or that $4n^2+3n+10 \sim n^2$. But how about more complicated functions? say $n^n + n! + n^{\log\log n} + n^{1/logn}$).

$\Downarrow$

- We want to express **rate of growth** of a function:
  - the dominant term with respect to n
  - ignoring constants in front of it

$k_1 n + k_2 \sim n$ $k_1 n \log n \sim n \log n$ $k_1 n^2 + k_2 n + k_3 \sim n^2$

- We also want to formalize that a e.g. $n \log n$ algorithm is better than a n^2 algorithm.

$\Downarrow$

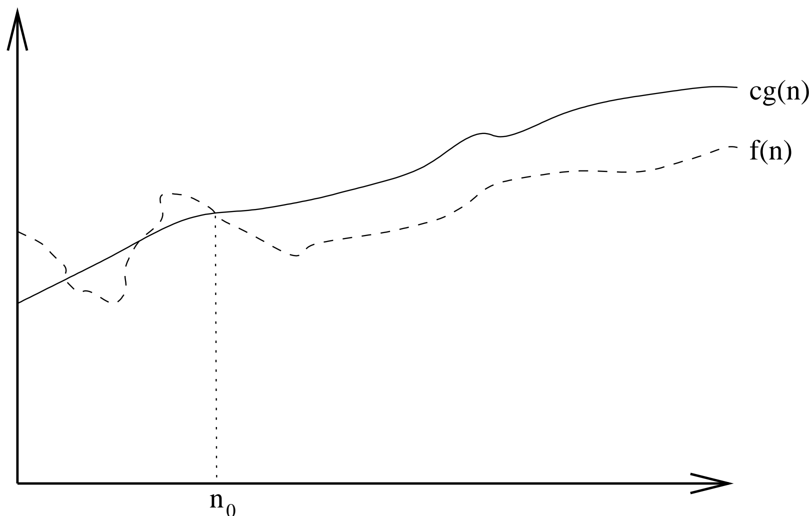
- O-notation (Big-O)
- Ω-notation
- Θ-notation
- you have probably seen it intuitively defined but we will now define it more carefully.

4.1 O-notation (Big-O)

$O(g(n)) = \{f(n) : \exists\ c, n_0 > 0 \text{ such that } f(n) \leq cg(n)\ \forall n \geq n_0\}$

- $O(\cdot)$ is used to asymptotically *upper bound* a function.

We think of $f(n) \in O(g(n))$ as corresponding to $f(n) \leq g(n)$.



Examples:

- $1/3n^2 - 3n \in O(n^2)$ because $1/3n^2 - 3n \leq cn^2$ if $c \geq 1/3 - 3/n$ which holds for $c = 1/3$ and $n > 1$.
- $k_1n^2 + k_2n + k_3 \in O(n^2)$ because $k_1n^2 + k_2n + k_3 < (k_1 + |k_2| + |k_3|)n^2$ and for $c > k_1 + |k_2| + |k_3|$ and $n \geq 1$, $k_1n^2 + k_2n + k_3 < cn^2$.
- $k_1n^2 + k_2n + k_3 \in O(n^3)$ as $k_1n^2 + k_2n + k_3 < (k_1 + k_2 + k_3)n^3$
- $f(n) = n^2/3 - 3n$, $g(n) = n^2$
  - $f(n) \in O(g(n))$
  - $g(n) \in O(f(n))$
- $f(n) = an^2 + bn + c$, $g(n) = n^2$
  - $f(n) \in O(g(n))$
  - $g(n) \in O(f(n))$
- $f(n) = 100n^2$, $g(n) = n^2$
  - $f(n) \in O(g(n))$
  - $g(n) \in O(f(n))$
- $f(n) = n, g(n) = n^2$
  - $f(n) \in O(g(n))$

Note: $O(\cdot)$ gives an upper bound of f, but not necessarilly tight:

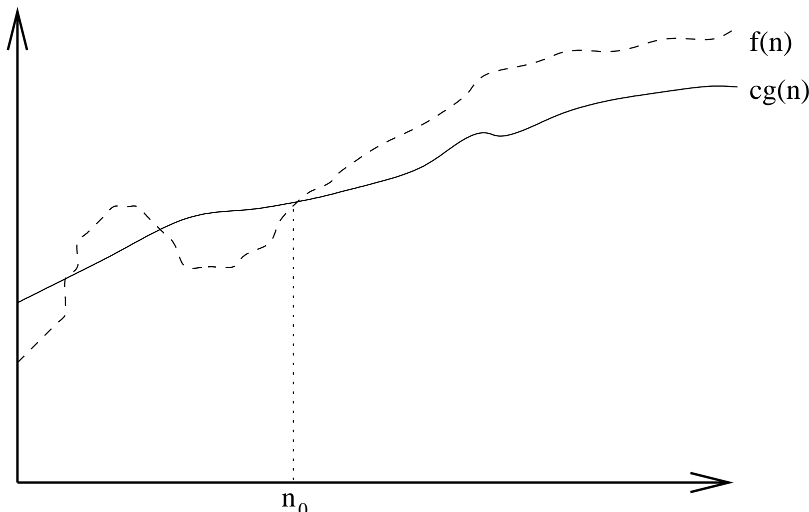
- $n \in O(n)$, $n \in O(n^2)$, $n \in O(n^3)$, $n \in O(n^{100})$

4.2 Ω-notation (big-Omega)

$\Omega(g(n)) = \{f(n) : \exists\ c, n_0 > 0 \text{ such that } cg(n) \leq f(n)\ \forall n \geq n_0\}$

- $\Omega(\cdot)$ is used to asymptotically *lower bound* a function.

We think of $f(n) \in \Omega(g(n))$ as corresponding to $f(n) \geq g(n)$.



Examples:

- $1/3n^2 - 3n \in \Omega(n^2)$ because $1/3n^2 - 3n \geq cn^2$ if $c \leq 1/3 - 3/n$ which is true if $c = 1/6$ and $n > 18$.
- $k_1n^2 + k_2n + k_3 \in \Omega(n^2)$.
- $k_1n^2 + k_2n + k_3 \in \Omega(n)$ (lower bound!)
- $f(n) = n^2/3 - 3n$, $g(n) = n^2$
  - $f(n) \in \Omega(g(n))$
  - $g(n) \in \Omega(f(n))$
- $f(n) = an^2 + bn + c$, $g(n) = n^2$
  - $f(n) \in \Omega(g(n))$
  - $g(n) \in \Omega(f(n))$
- $f(n) = 100n^2$, $g(n) = n^2$
  - $f(n) \in \Omega(g(n))$
  - $g(n) \in \Omega(f(n))$
- $f(n) = n, g(n) = n^2$
  - $g(n) \in \Omega(f(n))$

Note: $\Omega(\cdot)$ gives a lower bound of f, but not necessarily tight:

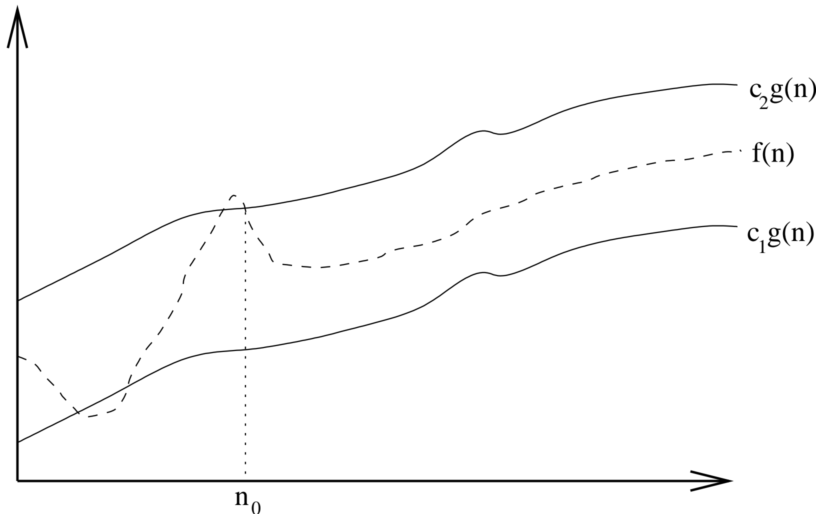
- $n \in \Omega(n)$, $n^2 \in \Omega(n)$, $n^3 \in \Omega(n)$, $n^{100} \in \Omega(n)$

4.3 Θ-notation (Big-Theta)

$\Theta(g(n)) = \{f(n) : \exists\ c_1, c_2, n_0 > 0 \text{ such that } c_1 g(n) \le f(n) \le c_2 g(n)\ \forall n \ge n_0\}$

- $\Theta(\cdot)$ is used to asymptotically *tight bound* a function.

We think of $f(n) \in \Theta(g(n))$ as corresponding to $f(n) = g(n)$.



$f(n) = \Theta(g(n))$ *if and only if* $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

It is easy to see (try it!) that:

Theorem 1 *If $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$ then $f(n) \in \Theta(g(n))$.*

Examples:

- $k_1 n^2 + k_2 n + k_3 \in \Theta(n^2)$
- *worst case* running time of insertion-sort is $\Theta(n^2)$
- $6n \log n + \sqrt{n} \log^2 n \in \Theta(n \log n)$:
  - We need to find n_0, c_1, c_2 such that $c_1 n \log n \le 6n \log n + \sqrt{n} \log^2 n \le c_2 n \log n$ for $n > n_0$
    $c_1 n \log n \le 6n \log n + \sqrt{n} \log^2 n \Rightarrow c_1 \le 6 + \frac{\log n}{\sqrt{n}}$. Ok if we choose $c_1 = 6$ and $n_0 = 1$.
    $6n \log n + \sqrt{n} \log^2 n \le c_2 n \log n \Rightarrow 6 + \frac{\log n}{\sqrt{n}} \le c_2$. Is it ok to choose $c_2 = 7$? Yes, $\log n \le \sqrt{n}$ if $n \ge 2$.
  - So $c_1 = 6$, $c_2 = 7$ and $n_0 = 2$ works.
- $n^2/3 - 3n \in O(n^2), n^2/3 - 3n \in \Omega(n^2) \longrightarrow n^2/3 - 3n \in \Theta(n^2)$
- $an^2 + bn + c \in O(n^2), an^2 + bn + c \in \Omega(n^2) \longrightarrow an^2 + bn + c \in \Theta(n^2)$
- $n \ne \Theta(n^2)$
- $f(n) = 6n \lg n + \sqrt{n} \lg^n$, $g(n) = n \lg n$

4.4 Comments

The correct way to say is that $f(n) \in O(g(n))$, $f(n) \in \Omega(g(n))$ or $f(n) \in Theta(g(n))$. Abusing notation, people normally write $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$, respectively.

$3n^2+2n+10 = O(n^2), an^2+bn+c = O(n^2), n = O(n^2), n^2 = \Omega(n), n \log n = \Omega(n), 2n^2+3n = \Theta(n^2)$

- When we say "the running time is $O(n^2)$" we mean that the worst-case running time is $O(n^2)$ — best case might be better.
- When we say "the running time is $\Omega(n^2)$", we mean that the *best case* running time is $\Omega(n^2)$ — the worst case might be worse.
- Insertion-sort:
  - Best case: $\Omega(n)$
  - Worst case: $O(n^2)$
  - We can also say worst case is $\Theta(n^2)$ because there exists an input for which insertion sort takes $\Omega(n^2)$.
  - But, we cannot say that the running time of insertion sort is $\Theta(n^2)$!!!
- Use of O-notation makes it much easier to analyze algorithms; we can easily prove the $O(n^2)$ insertion-sort time bound by saying that both loops run in $O(n)$ time.
- We often use $O(n)$ in equations and recurrences: e.g. $2n^2 + 3n + 1 = 2n^2 + O(n)$ (meaning that $2n^2 + 3n + 1 = 2n^2 + f(n)$ where $f(n)$ is some function in $O(n)$).
- We use $O(1)$ to denote constant time.
- One can also define o and ω
  - $f(n) = o(g(n))$ corresponds to $f(n) < g(n)$
  - $f(n) = \omega(g(n))$ corresponds to $f(n) > g(n)$
  - we will not use them; we'll aim for tight bounds Θ.
- Some properties:
  - $f = O(g) \longrightarrow g = \Omega(f)$
  - $f = \Theta(g) \longrightarrow g = \Theta(f)$
  - reflexivity: $f = O(f), f = \Omega(f), f = \Theta(f)$
  - transitivity: $f = O(g), g = O(h) \longrightarrow f = O(h)$ etc
- Not all functions are asymptotically comparable! There exist functions f, g such that f is not $O(g)$, f is not $\Omega(g)$ (and f is not $\Theta(g)$).

5 Review of Log and Exp

- Base 2 logarithm comes up all the time (from now on we will always mean $\log_2 n$ when we write $\log n$ or $\lg n$).
  - Number of times we can divide n by 2 to get to 1 or less.
  - Number of bits in binary representation of n.
  - Inverse function of $2^n = 2 \cdot 2 \cdot 2 \cdots 2$ (n times).
  - Way of doing multiplication by addition: $\log(ab) = \log(a) + \log(b)$
  - Note: $\log n << \sqrt{n} << n$
- Properties:
  - $\lg^k n = (\lg n)^k$
  - $\lg \lg n = \lg(\lg n)$
  - $a^{\log_b c} = c^{\log_b a}$
  - $a^{\log_a b} = b$
  - $\log_a n = \frac{\log_b n}{\log_b a}$
  - $\lg b^n = n \lg b$
  - $\lg xy = \lg x + \lg y$
  - $\log_a b = \frac{1}{\log_b a}$

6 Growth Rate of Standard Functions

- Polynomial of degree d:

$$p(n) = \sum_{i=1}^{d} a_i \cdot n^i = \Theta(n^d)$$

  where $a_1, a_2, \ldots, a_d$ are constants (and $a_d > 0$).
- Any polylog grows slower than any polynomial.

$$\log^a n = O(n^b), \forall a > 0$$

- Any polynomial grows slower than any exponential with base $c > 1$.

$$n^b = O(c^n), \forall b > 0, c > 1$$