Lecture 15: Greedy Algorithms

CLRS 16.1-16.2

June 7th, 2002

1 Greedy Algorithms

- We have previously discussed *dynamic programming*—a way of improving on inefficient divide-and-conquer algorithm:
  - If same subproblem is solved several times, use table to store result of a subproblem the first time it is computed and never compute it again.
  - Alternatively, we can think about filling up a table of subproblem solutions from the bottom.
- In divide-and-conquer (and thus dynamic programming) we used the fact that the solution to a problems depends on solutions to smaller subproblems.
- Another, simpler and often less powerful (and less well-defined), technique that uses the same feature is *greediness*
- Like in the case of dynamic programming, we will introduce greedy algorithms via an example.

1.1 Activity Selection

- Problem: Given a set $A = \{A_1, A_2, \cdots, A_n\}$ of n activities with start and finish times (s_i, f_i), $1 \leq i \leq n$, select maximal set S of "non-overlapping" activities.
  - One can think of the problem as corresponding to scheduling the maximal number of classes (given their start and finish times) in one classroom.
- Solution:
  - Sort activity by finish time (let $A_1, A_2, \cdots, A_n$ denote sorted sequence)
  - Pick first activity A_1
  - Remove all activities with start time before finish time of A_1
  - Recursively solve problem on remaining activities.

- Program:

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Sort A by finish time
S = {A_1}
j = 1
FOR i = 2 to n DO
    IF s_i ≥ s_j THEN
        S = S ∪ {A_i}
        j = i
    FI
OD
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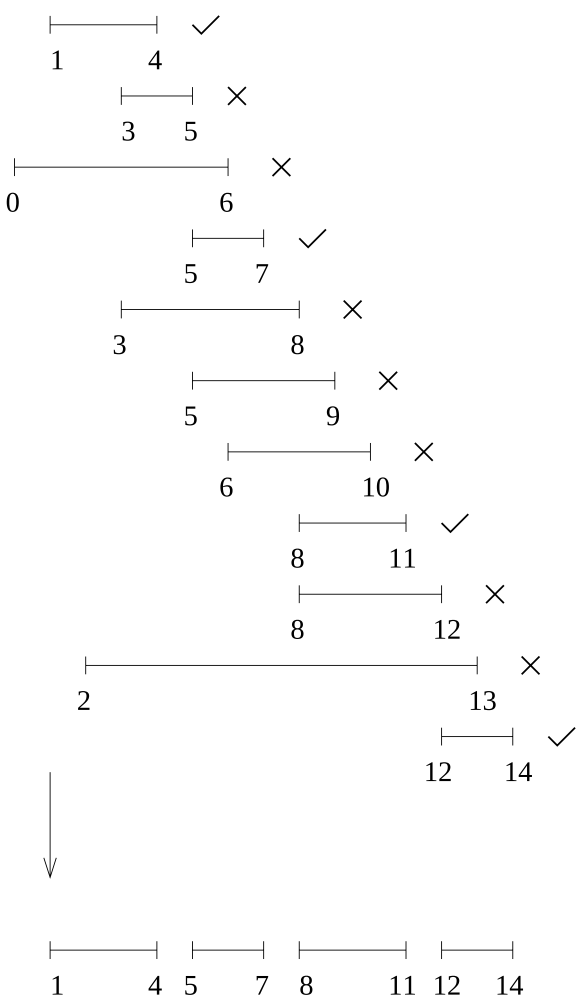
- Example:
  - 11 activities sorted by finish time: $(1, 4), (3, 5), (0, 6), (5, 7), (3, 8), (5, 9), (6, 10), (8, 11), (8, 12), (2, 13), (12, 14)$

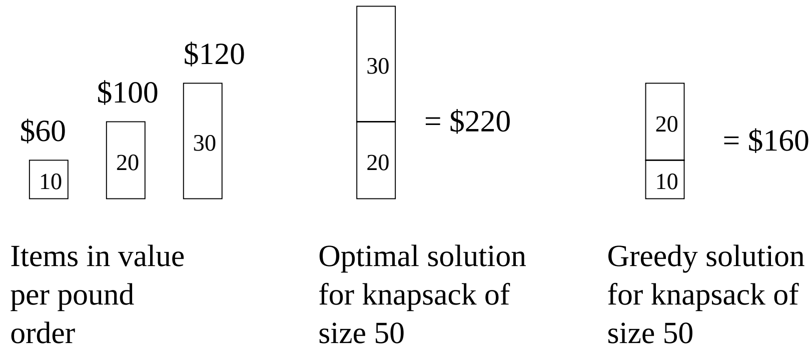


- Running time is obviously $O(n \log n)$.

- Is algorithm correct?
  - Output is set of non-overlapping activities, but is it the largest possible?
- Proof of correctness:
  - Given activities $A = \{A_1, A_2, \cdots, A_n\}$ ordered by finish time, there is an optimal solution containing A_1:
    - Suppose $S \subseteq A$ is optimal solution
    - If $A_1 \in S$, we are done
    - If $A_1 \notin S$:
      - Let first activity in S be A_k
      - Make new solution $S' = S \setminus \{A_k\} \cup \{A_1\}$ by removing A_k and using A_1 instead
      - S' is valid solution ($f_1 < f_k$) of maximal size ($|S'| = |S|$)
  - S is an optimal solution for A containing $A_1 \Rightarrow S' = S \setminus \{A_1\}$ optimal solution for $A' = \{A_i \in A : s_j \geq f_1\}$ (e.g. after choosing A_1 the problem reduces to finding optimal solution for activities not overlapping with A_1)
    - Suppose we have solution S'' to A' such that $|S''| > |S'| = |S| - 1$
    - $S''' = S'' \cup \{A_1\}$ would be solution to A
    - Contradiction since we would have $|S'''| > |S|$
  - Correctness follows by induction on size of S
- Comparison of greedy algorithm technique with dynamic programming (divide-and-conquer):
  - In greedy algorithm we choose what looks like best solution at any given moment and recurse (choice does not depend on solution to subproblems).
  - In dynamic programming, solution depends on solution to subproblems.
  - Both techniques use optimal solution to subproblems (optimal solution "contains optimal solution for subproblems within it").
- It is often hard to figure out when being greedy works!

  Example:
  - $0-1$ KNAPSACK PROBLEM: Given n items, with item i being worth \$ v_i and having weight w_i pounds, fill knapsack of capacity w pounds with maximal value.
  - FRACTIONAL KNAPSACK PROBLEM: As $0-1$ KNAPSACK PROBLEM but we can take fractions of items.
  - Problems appear very similar, but only FRACTIONAL KNAPSACK PROBLEM can be solved greedily:
    - Compute value per pound $\frac{v_i}{w_i}$ for each item
    - Sort items by value per pound.
    - Fill knapsack greedily (take objects in order)
      $\Downarrow$
      $O(n \log n)$ time, easy to show that solution is optimal.

– Example that $0-1$ KNAPSACK PROBLEM cannot be solved greedily:



Note: In FRACTIONAL KNAPSACK PROBLEM we can take $\frac{2}{3}$ of \$120 object and get \$240 solution.

- $0-1$ KNAPSACK PROBLEM can be solved in time $O(n \cdot w)$ using dynamic-programming (homework).