Lecture 14: Dynamic Programming

(CLRS 15.2-15.3)

June 6th, 2002

1 Dynamic programming

- We have previously discussed how divide-and-conquer can often be used to obtain efficient algorithms.
  - Examples: matrix multiplication, merge-sort, quick-sort,....
- Sometimes direct use of divide-and-conquer does not yield efficient algorithms—in fact, sometimes it results in really bad algorithms.
- Today we will discuss a technique which can often be used to improve upon an inefficient divide-and-conquer algorithm.
  - The technique is called "Dynamic programming". It is neither especially 'dynamic' nor especially 'programming' related.
  - We will discuss dynamic programming by looking at an example.

1.1 Matrix-chain multiplication

- Problem: Given a sequence of matrices $A_1, A_2, A_3, ..., A_n$, find the best way (using the minimal number of multiplications) to compute their product.
  - Isn't there only one way? $((\cdots((A_1 \cdot A_2) \cdot A_3) \cdots) \cdot A_n)$
  - No, matrix multiplication is *associative*.
    e.g. $A_1 \cdot (A_2 \cdot (A_3 \cdot (\cdots (A_{n-1} \cdot A_n) \cdots)))$ yields the same matrix.
  - Different multiplication orders do not cost the same:
    * Multiplying $p \times q$ matrix A and $q \times r$ matrix B takes $p \cdot q \cdot r$ multiplications; result is a $p \times r$ matrix.
    * Consider multiplying 10×100 matrix A_1 with 100×5 matrix A_2 and 5×50 matrix A_3.
      - $(A_1 \cdot A_2) \cdot A_3$ takes $10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50 = 7500$ multiplications.
      - $A_1 \cdot (A_2 \cdot A_3)$ takes $100 \cdot 5 \cdot 50 + 10 \cdot 50 \cdot 100 = 75000$ multiplications.
- In general, let A_i be $p_{i-1} \times p_i$ matrix.
  - $A_1, A_2, A_3, \ldots, A_n$ can be represented by $p_0, p_1, p_2, p_3, \ldots, p_n$
- Let $m(i, j)$ denote minimal number of multiplications needed to compute $A_i \cdot A_{i+1} \cdots A_j$
  - We want to compute $m(1, n)$.

- Divide-and-conquer solution/recursive algorithm:
  - Divide into $j-i-1$ subproblems by trying to set parenthesis in all $j-i-1$ positions. (e.g. $(A_i \cdot A_{i+1} \cdots A_k) \cdot (A_{k+1} \cdots A_j)$ corresponds to multiplying $p_{i-1} \times p_k$ and $p_k \times p_j$ matrices.)
  - Recursively find best way of solving sub-problems. (e.g. best way of computing $A_i \cdot A_{i+1} \cdots A_k$ and $A_{k+1} \cdot A_{k+2} \cdots A_j$)
  - Pick best solution.
- Algorithm expressed in terms of $m(i,j)$:

$$m(i,j) = \begin{cases} 0 & \text{If } i = j \\ \min_{i \le k < j}\{m(i,k) + m(k+1,j) + p_{i-1} \cdot p_k \cdot p_j\} & \text{If } i < j \end{cases}$$

- Program:

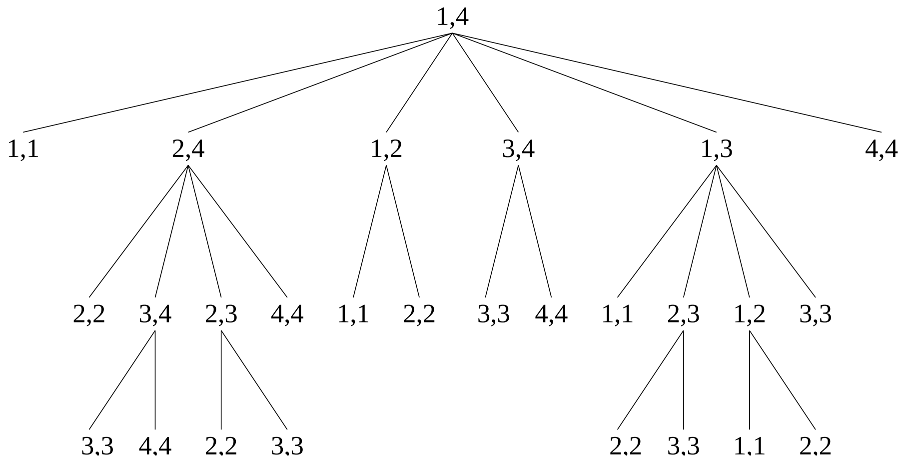
```
MATRIX-CHAIN(i, j)
    IF i = j THEN return 0
    m(i, j) = ∞
    FOR k = i TO j − 1 DO
        q = MATRIX-CHAIN(i, k) + MATRIX-CHAIN(k + 1, j) +p_{i−1} · p_k · p_j
        IF q < m(i, j) THEN m(i, j) = q
    OD
    Return m(i, j)
END MATRIX-CHAIN

Return MATRIX-CHAIN(1, n)
```

- Running time:

$$\begin{aligned} T(n) &= \sum_{k=1}^{n-1}(T(k) + T(n-k) + O(1)) \\ &= 2 \cdot \sum_{k=1}^{n-1} T(k) + O(n) \\ &\ge 2 \cdot T(n-1) \\ &\ge 2 \cdot 2 \cdot T(n-2) \\ &\ge 2 \cdot 2 \cdot 2 \ldots \\ &= 2^n \end{aligned}$$

- Problem is that we compute the same result over and over again.
  - Example: Recursion tree for MATRIX-CHAIN$(1,4)$



We for example compute MATRIX-CHAIN$(3, 4)$ twice

- Solution is to "remember" values we have already computed in a table—*memorization*

MATRIX-CHAIN(i, j)
    IF $i = j$ THEN return 0
    IF $m(i, j) < \infty$ THEN return $m(i, j)$ /* This line has changed */
    FOR $k = i$ to $j - 1$ DO
        q = MATRIX-CHAIN(i, k) + MATRIX-CHAIN$(k + 1, j)$+$p_{i-1} \cdot p_k \cdot p_j$
        IF $q < m(i, j)$ THEN $m(i, j) = q$
    OD
    return $m(i, j)$
END MATRIX-CHAIN

FOR $i = 1$ to n DO
    FOR $j = i$ to n DO
        $m(i, j) = \infty$
    OD
OD

return MATRIX-CHAIN$(1, n)$

- Running time:
  - $\Theta(n^2)$ different calls to MATRIX-CHAIN(i, j).
  - The first time a call is made it takes $O(n)$ time, *not* counting recursive calls.
  - When a call has been made once it costs $O(1)$ time to make it again.
$\Downarrow$
$O(n^3)$ time
  - Another way of thinking about it: $\Theta(n^2)$ total entries to fill, it takes $O(n)$ to fill one.

1.2 Alternative view of Dynamic Programming

- Often (including in the book) dynamic programming is presented in a different way; As filling up a table from the bottom.
- Matrix-chain example: Key is that $m(i,j)$ only depends on $m(i,k)$ and $m(k+1,j)$ where $i \leq k < j \Rightarrow$ if we have computed them, we can compute $m(i,j)$
  - We can easily compute $m(i,i)$ for all $1 \leq i \leq n$ $(m(i,i) = 0)$
  - Then we can easily compute $m(i,i+1)$ for all $1 \leq i \leq n-1$
    $m(i,i+1) = m(i,i) + m(i+1,i+1) + p_{i-1} \cdot p_i \cdot p_{i+1}$
  - Then we can compute $m(i,i+2)$ for all $1 \leq i \leq n-2$
    $m(i,i+2) = \min\{m(i,i) + m(i+1,i+2) + p_{i-1} \cdot p_i \cdot p_{i+2}, m(i,i+1) + m(i+2,i+2) + p_{i-1} \cdot p_{i+1} \cdot p_{i+2}\}$
    $\vdots$
  - Until we compute $m(1,n)$
  - Computation order:

j

i	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2		1	2	3	4	5	6
3			1	2	3	4	5
4				1	2	3	4
5					1	2	3
6						1	2
7							1

− Computation order

- Program:

FOR $i = 1$ to n DO
  $m(i,i) = 0$
OD
FOR $l = 1$ to $n-1$ DO
  FOR $i = 1$ to $n-l$ DO
    $j = i + l$
    $m(i,j) = \infty$
    FOR $k = 1$ to $j-1$ DO
      $q = m(i,k) + m(k+1,j) + p_{i-1} \cdot p_k \cdot p_j$
      IF $q < m(i,j)$ THEN $m(i,j) = q$
    OD
  OD
OD

- Analysis:
  - $O(n^2)$ entries, $O(n)$ time to compute each $\Rightarrow O(n^3)$.
- Note:
  - I like recursive (divide-and-conquer) thinking.
  - Book seems to like table method better.
  - I like divide-and-conquer because one does not need to get new idea (write new program)—just use table!