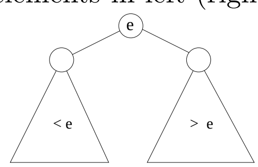
Lecture 13: Augmented Search Trees

(CLRS 14)

June 5th, 2002

1 Red-Black Trees

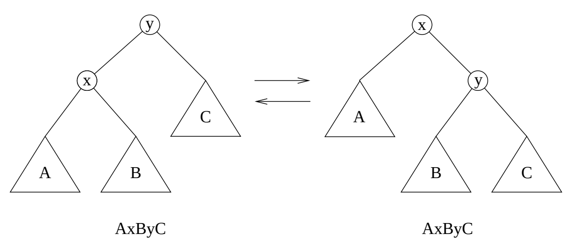
- Last time we discussed red-black trees:
  - Balanced binary trees—all elements in left (right) subtree of node x are $< x$ ($> x$).



  - Every node is colored RED or BLACK and we maintained red-blue invariant:

∗ Root is BLACK. ∗ A RED node can only have BLACK children. ∗ Every path from the root to a leaf contains the same number of BLACK nodes.

- We saw how the red-blue invariant guaranteed $O(\log n)$ height.
- We could reestablish the red-blue invariant after an insertion or deletion in $O(\log n)$ time
  - $O(\log n)$ node recolorings (no structural changes).
  - $O(1)$ rotations:

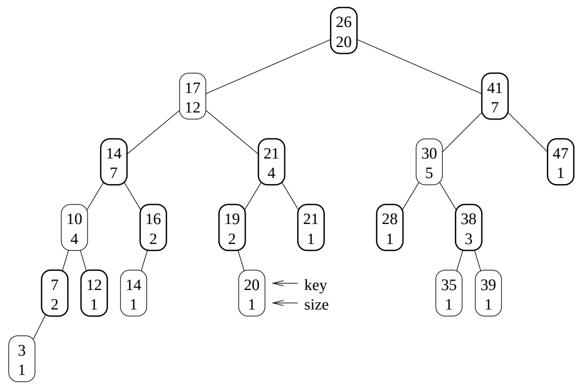


- Red-black tree also supports SEARCH, SUCCESSOR, and PREDECESSOR in $O(\log n)$ as in binary search trees.
- We will now discuss how to develop data structures supporting other operations by *augmenting* red-black tree.

2 Augmented Data Structures

- We want to add an operation SELECT(i) to a red-black tree
  - We have previously seen how to select the i'th element among n elements in $O(n)$ time.
  - Can we support it faster if we have the elements stored in a data structure?
  - We can of course support the operation in $O(1)$ time if we have the elements sorted in an array but what is we also want to be able to insert and delete elements?
- We augment every node x in red-black tree with a field $size(x)$ equal to the number of nodes in the subtree rooted in x
  - $size(x) = size(left(x)) + size(right(x)) + 1$

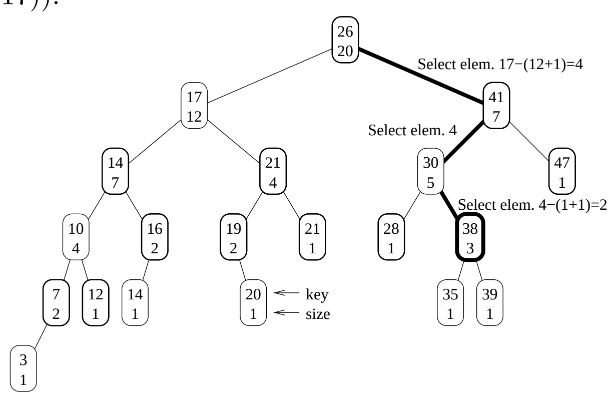
  Example:



- We can use this field to implement SELECT(i):

```
SELECT(x, i)
    r = size(left(x)) + 1
    IF i = r THEN Return x
    IF i < r THEN Return Select(left(x), i)
    IF i > r THEN Return Select(right(x), i−r)
```

Example (SELECT(17)):



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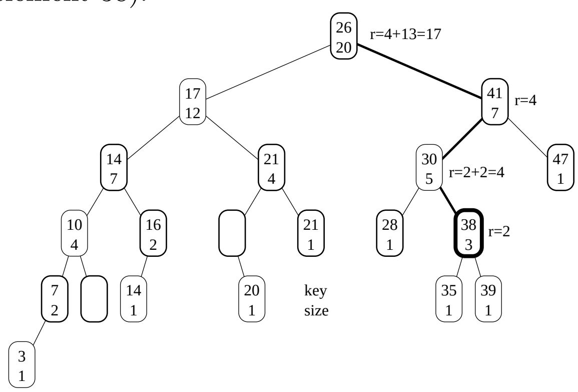
Since we only follow one root-leaf path the operation takes $O(\log n)$ time.

- Actually, we can also use the field to perform the "opposite" operation in $O(\log n)$ time—determining the *rank* of the element in node x:

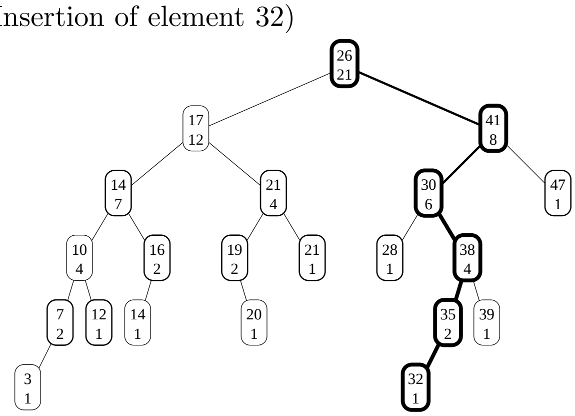
```
RANK(x)
    r = size(left(x)) + 1
    y = x
    WHILE y ≠ root of tree DO
        IF y = right(parent(y)) THEN
            r = r + size(left(parent(y))) + 1
            y = parent(y)
        FI
    OD
    Return r
```

Example (RANK of element 38):

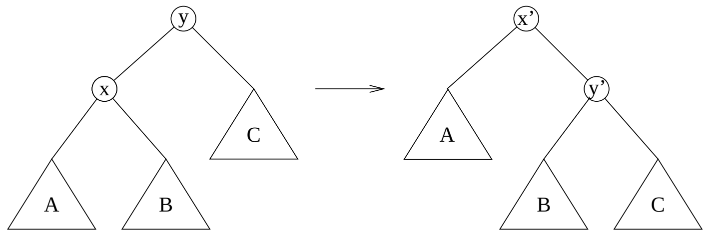


- We need to maintain the extra field during updates:
  - INSERT(i):
    - ∗ Search down one root-leaf part as usual for position where i should be inserted.
    - ∗ Increment $size(x)$ for all nodes x on root-leaf path (these are the *only* nodes for which the size field change).

    Example (Insertion of element 32)



    * Rebalancing using Red-black tree rules—recall that we do $O(\log n)$ recolorings and $O(1)$ rotations:
        · Color change rules do not affect extra field
        · Rotations do affect size extra fields but we can still easily perform a rotation in $O(1)$ time



$size(y') = size(root(B)) + size(root(C)) + 1$
$size(x') = size(y) = size(root(A)) + size(root(B)) + size(root(C)) + 2$

    $\Downarrow$
    INSERT performed in $O(\log n)$ time.

  - DELETE(i):
    * Find element to delete and decrement size field on one root-leaf path (recall that conceptually we always delete a node with at most one child).
    * Rebalance using rotations.

  $\Downarrow$
  DELETE performed in $O(\log n)$ time.

- Note: The key to maintaining the size field during updates is that the field of node x only depend on the field of the children of $x \Rightarrow$
  - Insertion or deletion only affect one root-leaf path.
  - Rotations can be handled in $O(1)$ time locally.

- In general we can easily prove the following:

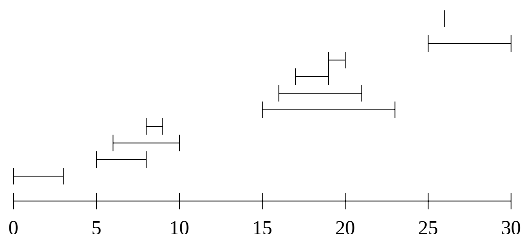
  A field f in a red-black tree can be maintained in $O(\log n)$ time during updates if $f(x)$ can be computed using only information in x, $left(x)$ and $right(x)$ (including $f(left(x))$ and $f(right(x))$)

  - When changing field in a node x, f can only change for the $O(\log n)$ ancestors of x on the path to the root.
  - Rotations can be handled in $O(1)$ time locally.

3 Interval Tree

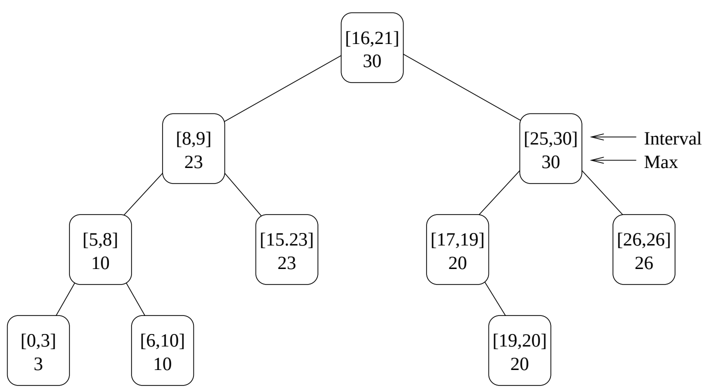
- We now consider a slightly more complicated augmentation. We want so solve the following problem:
  - Maintain a set of n intervals $[i_1, i_2]$ such that one of the intervals containing a query point q (if any) can be found efficiently.

  Example: A set of intervals. A query with $q = 9$ returns $[6, 10]$ or $[8, 9]$. A query with $q = 23$ returns $[15, 23]$.



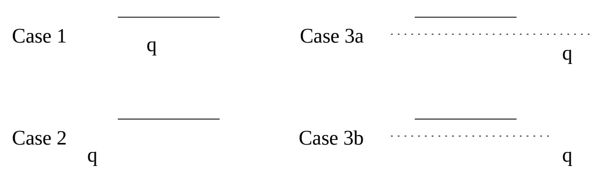
- To solve the problem we use the so-called "Interval tree":
  - Red-black tree with intervals in nodes
    * Key is left endpoint
  - Node x augmented with maximal right endpoint in subtree rooted in x

  Example: Interval tree on intervals from previous figure:



- We can maintain the interval tree dynamically during insertions and deletions in $O(\log n)$ time
  - because augmented field in x only depends on augmented fields in the children of x and the interval stored in x.
  - $max(x) = \max(rightendpoint(x), max(left(x)), max(right(x)))$

- We can also answer a query in $O(\log n)$ time:
  1. We first check if q is contained in interval stored in root r—if it is we are done.
  2. Next we check if q is on left side of left endpoint of interval in r—if it is we recursively search in left subtree (q cannot be contained in any interval in right subtree).
  3. If q is to the right of left endpoint of interval in r we have two cases:
     (a) If $max(left(r)) > q$ there must be a segment in left subtree containing q and we recurse left.
     (b) If $max(left(r)) < q$ there is no segment in left subtree containing q and we recurse right.




```
QUERY(x, q)
    IF q contained in x interval THEN Return x
    IF max(left(x)) ≥ q THEN
        Return Query(left(x), q)
    ELSE
        Return Query(right(x), q)
    FI
```

$\Downarrow$

We search down one root-leaf path $\Rightarrow O(\log n)$ time.

Example: Query with $q = 23$: