Lecture 2: Divide-and-Conquer and Growth of Functions

(CLRS 2.3,3)

May 17th, 2002

1 Designing Good Algorithms: Divide and Conquer/Mergesort

1.1 Divide-and-conquer

- Last time we discussed insertion sort
  - We introduced RAM model of computation and discussed its limitations.
  - We analyzed insertion sort in the RAM model
    - Best-case $k_1 n - k_2$.
    - Worst-case (and average case) $k_3 n^2 + k_4 - k_5$
  - We discussed how we are normally only interested in growth of running time:
    - Best-case *linear* in n ($\sim n$), worst-case *quadratic* in n ($\sim n^2$).
- Can we design better than n^2 sorting algorithm?
- We will do so using one of the most powerful algorithm design techniques.

Divide and Conquer

To Solve P:

1. *Divide* P into smaller problems $P_1, P_2, P_3 .....P_k$.
2. *Conquer* by solving the (smaller) subproblems recursively.
3. *Combine* solutions to $P_1, P_2, ...P_k$ into solution for P.

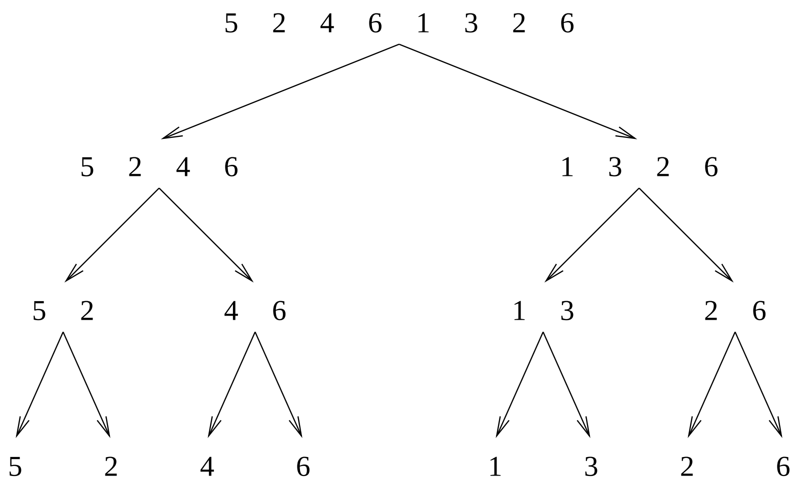
1.2 Merge-Sort

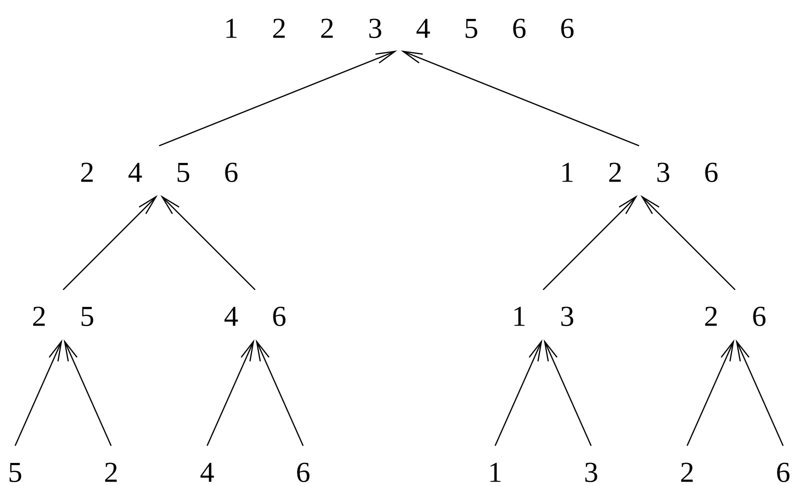
- Using divide-and-conquer, we can obtain a merge-sort algorithm.
  - Divide: Divide n elements into two subsequences of $n/2$ elements each.
  - Conquer: Sort the two subsequences recursively.
  - Combine: Merge the two sorted subsequences.
- Assume we have procedure Merge(A, p, q, r) which merges sorted A[p..q] with sorted A[q+1....r] in $(r - p)$ time.

- We can sort A[p...r] as follows (initially p=1 and r=n):

```
Merge Sort(A,p,r)
    If p < r then
        q = ⌊(p + r)/2⌋
        MergeSort(A,p,q)
        MergeSort(A,q+1,r)
        Merge(A,p,q,r)
```

Example:





1.3 Correctness

- Induction on n
  - Easy assuming Merge() is correct!

1.4 Analysis

- To simplify things, let us assume that n is a power of 2, i.e $n = 2^k$ for some k.
- Running time of the procedure can be analyzed using a recurrence equation/relation.

$$
\begin{aligned}
T(n) &\leq c_1 + T(n/2) + T(n/2) + c_2 n \\
&\leq 2T(n/2) + c_3 n
\end{aligned}
$$

$\Downarrow$

$T(n) \leq c_1 n \log_2 n$ as we will see later.

- We can also get $n \log_2 n$ bound by noticing that the recursion tree has depth $\log_2 n$ and that linear time is spent on each level.
- Note:
  - We really have $T(n) = c_4$ if $n = 1$
  - If $n \neq 2^k$ things can be complicated (We will often assume $n = 2^k$ to avoid complicated cases).

1.5 Log's

- Base 2 logarithm comes up all the time (from now on we will always mean $\log_2 n$ when we write $\log n$).
  - Number of times we can divide n by 2 to get to 1 or less.
  - Number of bits in binary representation of n.
  - Inverse function of $2^n = 2 \cdot 2 \cdot 2 \cdots 2$ (n times).
  - Way of doing multiplication by addition: $\log(ab) = \log(a) + \log(b)$
- Note:
  - $\log_a n = \frac{\log_b n}{\log_b a}$
  - $\log n << \sqrt{n} << n$

1.6 Algorithms matter!

- Sort 10 million integers on
  - 1 GHZ computer (1000 million instructions per second) using $2n^2$ algorithm.
  - 100 MHz computer (100 million instructions per second) using $50n \log n$ algorithm.
- Supercomputer : $\frac{2 \cdot (10^7)^2 \ inst.}{10^9 \ inst. \ per \ second} = 200000$ seconds ≈ 55 hours.
- Personal computer : $\frac{50 \cdot 10^7 \cdot \log 10^7 \ inst.}{10^8 \ inst. \ per \ second} < \frac{50 \cdot 10^7 \cdot 7 \cdot 3}{10^8} = 5 \cdot 7 \cdot 3 = 105$ seconds.

2 Asymptotic Growth

- In the insertion-sort example we discussed that when analyzing algorithms we are
  - interested in worst-case running time as function of input size n
  - not interested in exact constants in bound
  - not interested in lower order terms
- A good reason for not caring about constants and lower order terms is that the RAM model is not completely realistic anyway (not all operations cost the same)

$\Downarrow$

- We want to express *rate of growth* of standard functions:
  - the leading term with respect to n
  - ignoring constants in front of it

$$\boxed{\begin{array}{l} k_1 n + k_2 \sim n \\ k_1 n \log n \sim n \log n \\ k_1 n^2 + k_2 n + k_3 \sim n^2 \end{array}}$$

- We also want to formalize e.g. that a $n \log n$ algorithms is better than a n^2 algorithm.

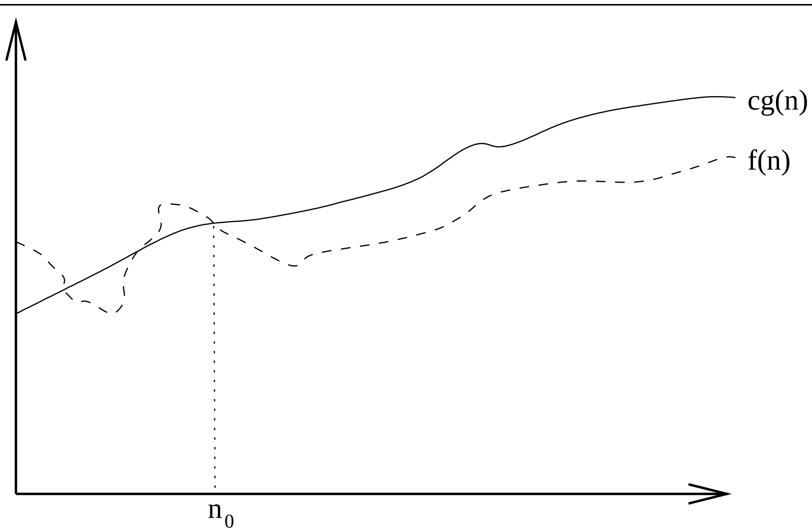
$\Downarrow$

- O-notation (Big-O)
  - you have probably all seen it intuitively defined but we will now define it more carefully.

2.1 O-notation (Big-O)

$O(g(n)) = \{f(n) : \exists\ c, n_0 > 0 \text{ such that } f(n) \leq cg(n)\ \forall n \geq n_0\}$

- $O(\cdot)$ is used to asymptotically *upper bound* a function.
- $O(\cdot)$ is used to bound *worst-case* running time.

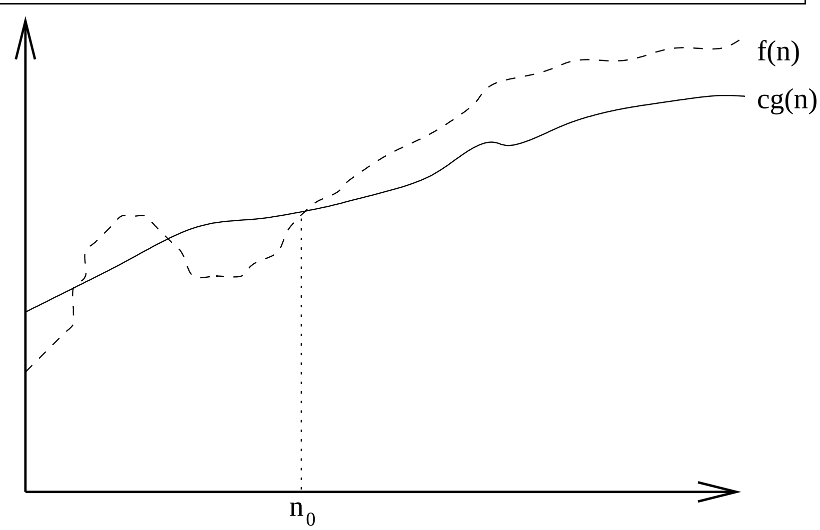


- Examples:
  - $1/3n^2 - 3n \in O(n^2)$ because $1/3n^2 - 3n \leq cn^2$ if $c \geq 1/3 - 3/n$ which holds for $c = 1/3$ and $n > 1$.
  - $k_1n^2 + k_2n + k_3 \in O(n^2)$ because $k_1n^2 + k_2n + k_3 < (k_1 + |k_2| + |k_3|)n^2$ and for $c > k_1 + |k_2| + |k_3|$ and $n \geq 1$, $k_1n^2 + k_2n + k_3 < cn^2$.
  - $k_1n^2 + k_2n + k_3 \in O(n^3)$ as $k_1n^2 + k_2n + k_3 < (k_1 + k_2 + k_3)n^3$ (Upper bound!).
- Note:
  - When we say "the running time is $O(n^2)$" we mean that the worst-case running time is $O(n^2)$ — best case might be better.
  - Use of O-notation often makes it much easier to analyze algorithms; we can easily prove the $O(n^2)$ insertion-sort time bound by saying that both loops run in $O(n)$ time.
  - We often abuse the notation a little:
    * We often write $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$.
    * We often use $O(n)$ in equations: e.g. $2n^2 + 3n + 1 = 2n^2 + O(n)$ (meaning that $2n^2 + 3n + 1 = 2n^2 + f(n)$ where $f(n)$ is some function in $O(n)$).
    * We use $O(1)$ to denote constant time.

2.2 Ω-notation (big-Omega)

$\Omega(g(n)) = \{f(n) : \exists\, c, n_0 > 0 \text{ such that } cg(n) \leq f(n)\ \forall n \geq n_0\}$

- $\Omega(\cdot)$ is used to asymptotically *lower bound* a function.



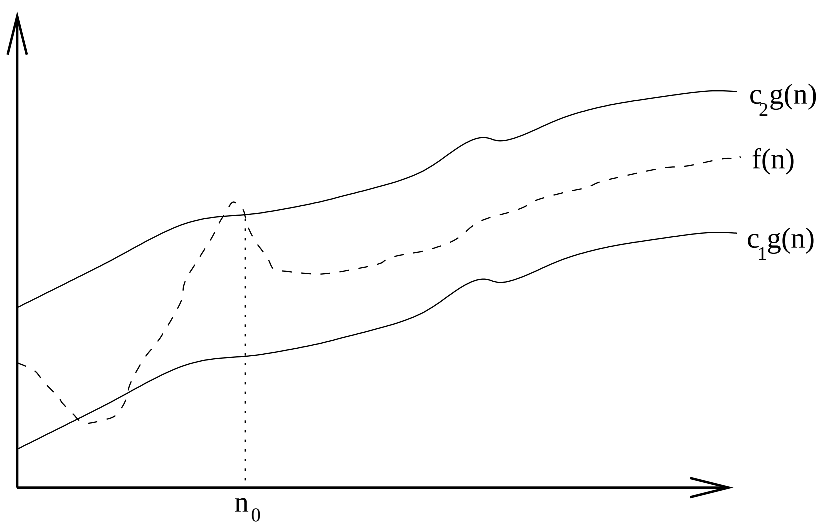
- Examples:
  - $1/3n^2 - 3n = \Omega(n^2)$ because $1/3n^2 - 3n \geq cn^2$ if $c \leq 1/3 - 3/n$ which is true if $c = 1/6$ and $n > 18$.
  - $k_1n^2 + k_2n + k_3 = \Omega(n^2)$.
  - $k_1n^2 + k_2n + k_3 = \Omega(n)$ (lower bound!)

- Note:
  - When we say "the running time is $\Omega(n^2)$", we mean that the *best case* running time is $\Omega(n^2)$ — the worst case might be worse.
- Insertion-sort:
  - Best case: $\Omega(n)$
  - Worst case: $O(n^2)$
  - We can also say that the *worst case* running time is $\Omega(n^2) \Rightarrow$ worst case running time is "precisely" n^2.

2.3 Θ-notation (Big-Theta)

$\Theta(g(n)) = \{f(n) : \exists\ c_1, c_2, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n)\ \forall n \geq n_0\}$

- $\Theta(\cdot)$ is used to asymptotically *tight bound* a function.



$f(n) = \Theta(g(n))$ *if and only if* $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

- Examples:
  - $k_1 n^2 + k_2 n + k_3 = \Theta(n^2)$
  - *worst case* running time of insertion-sort is $\Theta(n^2)$
  - $6n \log n + \sqrt{n} \log^2 n = \Theta(n \log n)$:
    - We need to find n_0, c_1, c_2 such that $c_1 n \log n \leq 6n \log n + \sqrt{n} \log^2 n \leq c_2 n \log n$ for $n > n_0$
      $c_1 n \log n \leq 6n \log n + \sqrt{n} \log^2 n \Rightarrow c_1 \leq 6 + \frac{\log n}{\sqrt{n}}$. Ok if we choose $c_1 = 6$ and $n_0 = 1$.
      $6n \log n + \sqrt{n} \log^2 n \leq c_2 n \log n \Rightarrow 6 + \frac{\log n}{\sqrt{n}} \leq c_2$. Is it ok to choose $c_2 = 7$? Yes, $\log n \leq \sqrt{n}$ if $n \geq 2$.
    - So $c_1 = 6$, $c_2 = 7$ and $n_0 = 2$ works.

- Note:
  - We often think of $f(n) = O(g(n))$ as corresponding to $f(n) \leq g(n)$.
  - Similarly, $f(n) = \Theta(g(n))$ corresponds to $f(n) = g(n)$
  - Similarly, $f(n) = \Omega(g(n))$ corresponds to $f(n) \geq g(n)$
  - One can also define o and ω
    * $f(n) = o(g(n))$ corresponds to $f(n) < g(n)$
    * $f(n) = \omega(g(n))$ corresponds to $f(n) > g(n)$

2.4 Growth rate of standard functions

- Book introduces standard functions in section 2.2 (we will introduce them as we need them):
  - Polynomial of degree d: $p(n) = \sum_{i=1}^{d} a_i \cdot n^i$ where $a_1, a_2, \ldots, a_d$ are constants (and $a_d > 0$). $p(n) = \Theta(n^d)$
- "Growth order": $\log\log n, \log n, \sqrt{n}, n, n \log\log n, n \log n, n \log^2 n, n^2, n^3, 2^n$
  - Growth rate of polynomials versus exponentials: $\lim_{n\to\infty} \frac{n^b}{a^n} = 0$.