# CPS 130 Homework 4 - Solutions

1. (CLRS 4.2-2) Argue that the solution to the recurrence

$$T(n) = T(n/3) + T(2n/3) + n$$

   is $\Omega(n \lg n)$ by appealing to a recursion tree.

   **Solution:** The recursion tree for the recurrence is given in CLRS. Note that the values across the levels of the recursion tree sum to $n$. The shortest path from a root to a leaf is

$$n \to \frac{1}{3}n \to \left(\frac{1}{3}\right)^2 n \to \ldots \to 1.$$

   Since $(1/3)^k n = 1$ when $k = \log_3 n$, the height of the tree is at least $\log_3 n$. Thus the solution to the recurrence is at least $n \log_3 n = \Omega(n \lg n)$.

2. Give asymptotic upper and lower bounds for the following recurrences. Assume T(n) is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers.

   - $T(n) = T(n-1) + n$

     **Solution:** Note that you cannot apply Master method. Assume $T(1) = 1$ and iterate:

$$\begin{aligned}
T(n) &= T(n-1) + n \\
&= T(n-2) + (n-1) + n \\
&= \ldots \\
&= T(n-1) + (n-1+1) + \ldots + n \\
&= T(1) + 2 + 3 + \ldots n \\
&= \sum_{i=1}^{n} i \\
&= \frac{n(n+1)}{2} \\
&= \Theta(n^2)
\end{aligned}$$

   - $T(n) = T(\sqrt{n}) + 1$

     **Solution:** Note that you cannot apply Master method. Assume $T(2) = 1$ and iterate:

$$\begin{aligned}
T(n) &= T(n^{\frac{1}{2}}) + 1 \\
&= T(n^{\frac{1}{4}}) + 1 + 1 \\
&= \ldots \\
&= T(n^{\frac{1}{2^i}}) + i
\end{aligned}$$

For this type of recurrence we cannot stop at 1 because $n^{\frac{1}{2^i}}$ cannot be 1, except at the limit. The recursion depth is given by $n^{\frac{1}{2^i}} = 2$, which gives $i = \log\log n$. Substituting in the relation above we get:

$$\begin{aligned} T(n) &= T(2) + \log\log n \\ &= \Theta(\log\log n) \end{aligned}$$

- $T(n) = 2T(n/2) + n/\lg n$
  **Solution:** Note that you cannot apply Master method. Assume $T(1) = 1$ and iterate:

$$\begin{aligned} T(n) &= 2T(\frac{n}{2}) + \frac{n}{\log n} \\ &= 2^2T(\frac{n}{2^2}) + 2 \cdot \frac{\frac{n}{2}}{\log\frac{n}{2}} \\ &= \ldots \\ &= 2^iT(\frac{n}{2^i}) + \sum_{k=0}^{i-1} \frac{n}{\log\frac{n}{2^k}} \\ &= 2^iT(\frac{n}{2^i}) + \sum_{k=0}^{i-1} \frac{n}{\log n - k} \end{aligned}$$

The recursion depth is $i = \log_2 n$ and substituting we get:

$$\begin{aligned} T(n) &= 2^{\log_2 n}T(1) + \sum_{k=0}^{\log_2 n - 1} \frac{n}{\log n - k} \\ &= nT(1) + n \cdot \sum_{k=0}^{\log_2 n - 1} \frac{1}{\log n - k} \\ &= nT(1) + n \cdot \sum_{k=1}^{\log_2 n} \frac{1}{k} \\ &= nT(1) + n \cdot H_{\log n} \\ &= nT(1) + n \cdot \Theta(\log\log n) \\ &= \Theta(n \log\log n) \end{aligned}$$

Here we used the standard notation for the harmonic sum

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)$$

- $T(n) = T(n-1) + 1/n$
  **Solution:** Assume $T(1) = 1$ and iterate:

$$\begin{aligned} T(n) &= T(n-1) + \frac{1}{n} \\ &= T(n-2) + \frac{1}{n-1} + \frac{1}{n} \\ &= \ldots \\ &= T(n-i) + \frac{1}{n-i+1} + .. + \frac{1}{n} \end{aligned}$$

Taking $i = n - 1$ we get:

$$\begin{aligned} T(n) &= T(1) + \frac{1}{2} + ... + \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{1}{k} \\ &= H_n = \Theta(\log n) \end{aligned}$$