CPS 130 Homework 3 - Solutions

1. (CLRS 3-2) (a) and (b) only. Indicate, for each pair of expressions (A, B) below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\epsilon > 0$ and $c > 1$ are constants. Your answer should be in the form of a table with 'yes' or 'no' written in each box:

$$(\lg^k n, n^\epsilon), (n^k, c^n), (\sqrt{n}, n^{\sin n}), (2^n, 2^{n/2}), (n^{\lg m}, m^{\lg n}), (\lg(n!), \lg(n^n)).$$

Solution: For (a)-(f):

	(A, B)	O	o	Ω	ω	Θ
a.	$(lg^k n, n^\epsilon)$	yes	yes	no	no	no
b.	(n^k, c^n)	yes	yes	no	no	no
c.	$(\sqrt{n}, n^{\sin n})$	no	no	no	no	no
d.	$(2^n, 2^{n/2})$	no	no	yes	yes	no
e.	$(n^{\lg m}, m^{\lg n})$	yes	no	yes	no	yes
f.	$(\lg(n!), \lg(n^n))$	yes	no	yes	no	yes

2. (CLRS A.1-1) Find a simple formula for $\sum_{k=1}^{n}(2k-1)$.

Solution:

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1) &= 2\sum_{k=1}^{n} k - n \\
&= 2\frac{n(n+1)}{2} - n \\
&= n^2 + n - n \\
&= n^2.
\end{aligned}$$

3. Prove by induction that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution: For the base case $n = 1$,

$$\sum_{i=1}^{n=1} i^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6} = \frac{n(n+1)(2n+1)}{6}.$$

Assume that the statement is true for some k and use this to show $k+1$:

$$\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \ldots = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Let $P(n)$ denote

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Since we proved the statement $P(1)$ and showed that the inductive hypothesis $P(k) \Rightarrow P(k+1)$, $P(n)$ is true for all $n \geq 1$.

4. Solve the recurrence: $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1) + n(n-1) & \text{if } n \geq 2 \end{cases}$

Hint: use $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution: By iteration:

$$\begin{aligned} T(n) &= T(n-2) + (n-1)(n-2) + n(n-1) \\ &= ... \\ &= T(1) + \sum_{i=1}^{n} i(i-1) \\ &= 1 + \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i \\ &= 1 + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \\ &= 1 + \frac{n(n^2-1)}{3} \\ &= \Theta(n^3) \end{aligned}$$

Comments: If you find the solution of a recurrence by iteration (or master method) you do not need to prove it by induction. If you guess the solution, then you must prove it by induction.