CPS 130 Homework 12
Red-Black Trees

Write and justify your answers in the space provided.[1]

1. (CLRS 13.1-5) Show that the longest simple path from a node x in a red-black tree to a descendant leaf has length at most twice that of the shortest simple path from node x to a descendant leaf

2. (CLRS 13.1-6) What is the largest possible number of internal nodes in a red-black tree with black-height k? What is the smallest possible number?

[1] Collaboration is allowed, even encouraged, provided that the names of the collaborators are listed along with the solutions. Students must write up the solutions on their own.

3. (CLRS 13-2) The **join** operation takes two dynamic sets S_1 and S_2 and an element x such that for any $x_1 \in S_1$ and $x_2 \in S_2$, we have $key[x_1] \leq key[x] \leq key[x_2]$. It returns a set $S = S_1 \cup \{x\} \cup S_2$. In this problem, we investigate how to implement the join operation on red-black trees.

   (a) Given a red-black tree T, we store its black-height as the field $bh[T]$. Argue that this field can be maintained by RB-INSERT and RB-DELETE without requiring extra storage in the tree and without increasing the asymptotic running times. Show while descending through T, we can determine the black-height of each node we visit in $O(1)$ time per node visited.

   We wish to implement the operation RB-JOIN(T_1, x, T_2) which destroys T_1 and T_2 and returns a red-black tree $T = T_1 \cup \{x\} \cup T_2$. Let n be the total number of nodes in T_1 and T_2.

   (b) Assume without loss of generality that $bh[T_1] \geq bh[T_2]$. Describe an $O(\lg n)$ time algorithm that finds a black node y in T_1 with the largest key from among those nodes whose black-height is $bh[T_2]$.

   (c) Let T_y be the subtree rooted at y. Describe how T_y can be replaced by $T_y \cup \{x\} \cup T_2$ in $O(1)$ time without destroying the binary-search-tree property.

   (d) What color should we make x so that red-black properties $1, 2,$ and 4 are maintained? Describe how property 3 can be enforced in $O(\lg n)$ time.

   (e) Argue that the running time of RB-JOIN is $O(\lg n)$