Summations

Module 2: Algorithm Analysis

1 Overview

We introduce the two basic summations that come up most often in the analysis of algorithms: arithmetic and geometric summations.

Goals:

- Understand arithmetic and geometric summations and their $\Theta()$ bound
- Recognize arithmetic and geometric summations in different forms and give $\Theta()$ bounds

2 Why summations?

- Summations come frequently in the analysis of algorithms
- For e.g. the running time of a *while* loop can be expressed as the sum of the running time of each iteration.
- We'll see that summations come up in analyzing recursive algorithms
- Let's analyze Insertion Sort Worst-case: Worst-case happens when each next element is smaller than all elements in hand and has to be inserted all the way at the beginning. This corresponds to an array sorted in decreasing order. The inner loop runs in $\Theta(i)$ time when inserting element i. The total worst-case running time is $\Theta(1) + \Theta(2) + \Theta(3) + ... + \Theta(n-1)$. Let's ignore the $\Theta()$ for now and write it as: $1 + 2 + 3 + ... + (n-1)$. This is a summation. We need to find it's order of growth.
  - Note that it's easy to argue that $1 + 2 + ... + (n-1) = O(n^2)$. Right? But is this a tight bound? It's not obvious and we need to deeper into the sum in order to see

3 The arithmetic sum: $1 + 2 + 3 + ... + n$

The arithmetic sum is the following: $1 + 2 + 3 + \cdots + n$. This sum can be computed for any n using a simple formula, and the credit for finding this formula goes to Gauss. (Note: you may have heard the story... if not, google it).

$$\boxed{1 + 2 + 3 + \cdots + ... + n = \frac{n(n+1)}{2}, \text{ for any n } \in N}$$

This is an old, simple and beautiful result, so take a moment and think about it. Plug in a value for n, for e.g. $n = 1000$. The formula says that $1 + 2 + 3 + ... + 1000 = 1000 \cdot 1001/2 = 500 \cdot 1001 = 500500$. You computed the sum of 1000 numbers with one addition, one multiplication and one division. That's pretty cool.

Since we are in an algorithms class, and we wear our algorithms hat, we don't need to know the exact formula, only it's order of growth. Since $n \cdot (n+1)/2 = \Theta(n^2)$, we get that:

$$\boxed{1+2+3+\cdots+n = \Theta(n^2), \text{ for any n } \in N}$$

4 The geometric sum: $1 + x + x^2 + x^3 + ... + x^n$

Let x be an arbitrary value ($x > 0, x \neq 1$). The geometric sum is the following: $1 + x + x^2 + \cdots x^n$. There exists a formula for this sum as well. It can be shown that:

$$\boxed{1 + x + x^2 + \cdots x^n = \tfrac{x^{n+1}-1}{x-1}, \text{ for any } x > 0, x \neq 1, \text{ for any n } \in N}$$

Examples:

- For e.g. let $x = 2$. The sum $1 + 2 + 2^2 + 2^3 + ... + 2^n$ is a geometric sum. Its value can be computed using the formula for any $n \in N$. $1 + 2 + 2^2 + 2^3 + ... + 2^n = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1$.
- Compute $1 + 2 + 4 + 8 + 16 + ... + 2^{10} =?$

  Answer: This is a geometric sum with $x = 2$ and $n = 10$. We get that $1+2+2^2+2^3+....+2^n = 2^{11} - 1 = 2047$
- Compute $1 + 5 + 25 + 125 + ... + 5^8 =?$

  Answer: This is a geometric sum with $x = 5$ and $n = 8$. We get that $1+5+25+125+....+5^8 = (5^9 - 1)/(5 - 1)$
- Compute $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^{10}} =?$

  Answer: This is a geometric sum with $x = \frac{1}{2}$ and $n = 10$. We get that $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^{10}} = (1/2^{11} - 1)/(1/2 - 1) = 2 \cdot (1 - 1/2^{11})$

Again, our algorithms hat comes to simplify our life: we are not interested in the exact formula for the sum, only in the order of growth. What is the order of growth for x^{n+1}? This actually depends on whether $x < 1$ or $x > 1$. Consider 2^n versus $\frac{1}{2}^n$. The first one is an exponential $2^n = \Theta(2^n)$, and the second one becomes smaller and smaller as n gets larger, going towards 0 in the limit. We get the following two cases depending on whether $x < 1$ or $x > 1$:

$$\boxed{1 + x + x^2 + ... + x^n = \tfrac{1-x^{n+1}}{1-x} = \Theta(1), \text{ for any } x < 1, \text{ for any } n \in N}$$

$$\boxed{1 + x + x^2 + ... + x^n = \tfrac{x^{n+1}-1}{x-1} = \Theta(x^n), \text{ for any } x > 1, \text{ for any } n \in N}$$

Perhaps you wondered why not $\Theta(x^{n+1})$ rather than $\Theta(x^n)$? Note that x is a constant, so the two are equivalent. $\Theta(x^n) = \Theta(x^{n+1})$.

Examples:

- Find a tight bound for $1 + \frac{1}{2} + \dots \frac{1}{2^n}$

  Answer: It's a geometric sum with $x = \frac{1}{2}$, therefore $1 + \frac{1}{2} + \dots \frac{1}{2^n} = \Theta(1)$.

- Find a tight bound for $1 + 3 + \dots 3^n$

  Answer: It's a geometric sum with $x = 3$, therefore $1 + 3 + 9 + \dots 3^n = \Theta(3^n)$.

5 Examples: Using summations in practice

When analyzing algorithms we'll often encounter arithmetic and geometric summations. You need to be able to recognize them and find out the corresponding $\Theta()$ bound.

Examples: Find a tight bound for the following summations:

1. $1 + 2 + 3 + ... + \lg n$

   Answer: Arithmetic sum with $n' = \lg n$, so we get $\Theta(n'^2) = \Theta((\lg n)^2) = \Theta(\lg^2 n)$

2. $1 + 2 + 3 + 4 + ... + n^2$

   Answer: Arithmetic sum with $n' = n^2$, so we get $\Theta(n'^2) = \Theta(n^4)$

3. $1 + 1/5 + 1/5^2 + ... + 1/5^n$

   Answer: Geometric sum with $x = \frac{1}{5}$, so we get $\Theta(1)$

4. $\frac{2}{3} + (\frac{2}{3})^2 + ... + (\frac{2}{3})^{\lg n}$

   Answer: Geometric sum with $x = \frac{2}{3}$, so we get $\Theta(1)$

5. $1 + 2 + 2^2 + 2^3 + ...2^n$

   Answer: Geometric sum with $x = 2$, so we get $\Theta(2^n)$

6. $n + \frac{n}{2} + \frac{n}{4} + ...$

   Answer: Note that this is the same as $n \cdot (1 + \frac{1}{2} + \frac{1}{4} + ...) = n \cdot \Theta(1) = \Theta(n)$

7. $n + \frac{2n}{3} + \frac{4n}{9} + ...$

   Answer: This is the same as $n \cdot (1 + \frac{2}{3} + \frac{4}{9} + ...) = n \cdot \Theta(1) = \Theta(n)$

Self Quiz

Find a tight bound for the following summations:

1. $1 + 10 + 10^2 + 10^3 + ...10^n$

2. $1 + 2 + 3 + ... + \frac{n}{2}$

3. $n + n/5 + n/5^2 + ... + n/5^n$

4. $1 + 2 + 3 + 4 + ... + (2n)$

Appendix

Quiz answers:

1. $\Theta(10^n)$
2. $\Theta(n^2)$
3. $\Theta(n)$
4. $\Theta(n^2)$