csci 210: Data Structures

Program Analysis

Summary

- Summary
  - analysis of algorithms
  - asymptotic analysis
    - big-O
    - big-Omega
    - big-theta
  - asymptotic notation
  - commonly used functions
  - discrete math refresher

- READING:
  - GT textbook chapter 4

Analysis of algorithms

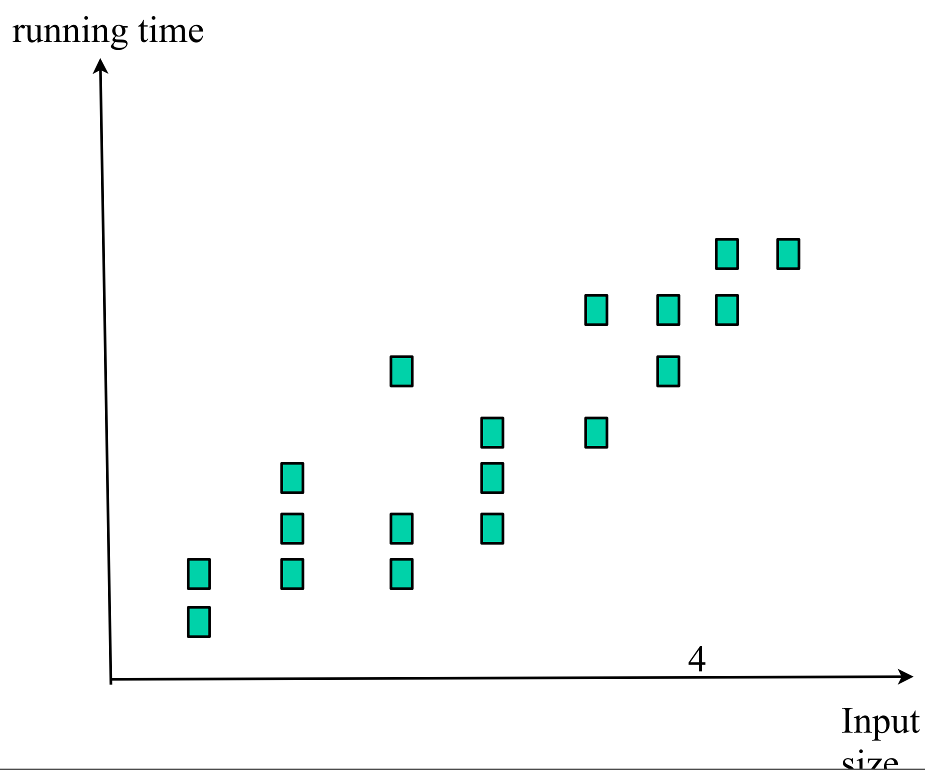
- Analysis of algorithms and data structure is the major force that drives the design of solutions.
  - there are many solutions to a problem
  - pick the one that is the most efficient
  - how to compare various algorithms? Analyze algorithms.
- Algorithm analysis: analyze the cost of the algorithm
  - cost = time: How much time does this algorithm require?
  - The primary efficiency measure for an algorithm is time
    - all concepts that we discuss for time analysis apply also to space analysis
  - cost = space: How much space (i.e. memory) does this algorithm require?
  - cost = space + time
  - etc
- Running time of an algorithm
  - increases with input size
  - on inputs of same size, can vary from input to input
    - e.g.: linear search an un-ordered array
  - depends on hardware
    - CPU speed, hard-disk, caches, bus, etc
  - depends on OS, language, compiler, etc

Analysis of algorithms

- Everything else being equal
  - we'd like to compare between algorithms
  - we'd like to study the relationship running time vs. size of input

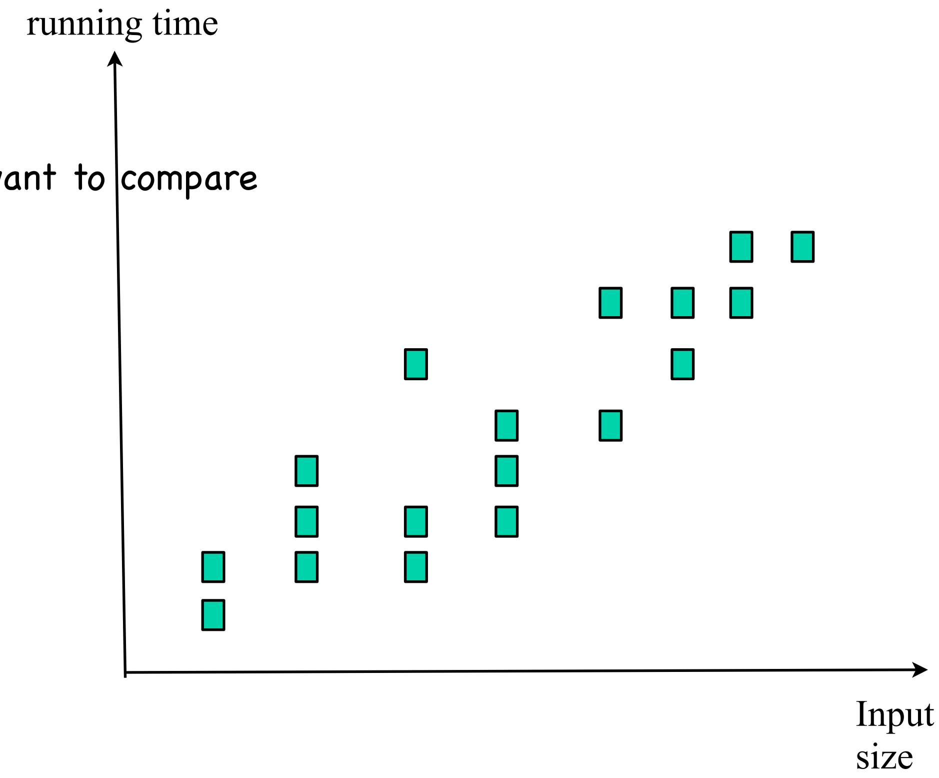
- How to measure running time of an algorithm?
  - 1. experimental studies
  - 2. theoretical analysis

- Experimental analysis
  - implement
  - chose various input sizes
  - for each input size, chose various inputs
    - run algorithm
    - time
    - compute average
    - plot



Experimental analysis

- Limitations
  - need to implement the algorithm
    - need to implement all algorithms that we want to compare
  - need many experiments
  - try several platforms
- Advantages
  - find the best algorithm in practice



- We would like to analyze algorithms without having to implement them
- Basically, we would like to be able to look at two algorithms flowcharts and decide which one is better

Theoretical analysis

- Model: RAM model of computation
  - Assume all operations cost the same
  - Assume all data fits in memory

- Running time (efficiency) of an algorithm:
  - the number if operations executed by the algorithm

- Does this reflect actual running time?
  - multiply nb. of instructions by processor speed
    - 1GHz processor ==> 10^9 instructions/second

- Is this accurate?
  - Not all instructions take the same...
  - various other effects.
  - Overall, it is a very good predictor of running time in most cases.

Notations

- Notation:
  - n = size of the input to the problem
- Running time:
  - number of operations/instructions on an input of size n
  - expressed as function of n: f(n)

- For an input of size n, running time may be smaller on some inputs than on others

- Best case running time:
  - the smallest number of operations on an input of size n
- Worst-case running time:
  - the largest number of operations on an input of size n
- For any n
  - best-case running time(n) <= running time(n) <= worst-case running time (n)

- Ideally, want to compute average-case running time
  - hard to model

Running times

- Expressed as functions of n: f(n)
- The most common functions for running times are the following:

  - constant time :
    - $f(n) = c$
  - logarithmic time
    - $f(n) = \lg n$
  - linear time
    - $f(n) = n$
  - n lg n
    - $f(n) = n \lg n$
  - quadratic
    - $f(n) = n^2$
  - cubic
    - $f(n) = n^3$
  - exponential
    - $f(n) = a^n$

Constant time

- $f(n) = c$
  - Meaning: for any n, $f(n)$ is a constant c

- Elementary operations
  - arithmetic operations
  - boolean operations
  - assignment statement
  - function call
  - access to an array element a[i]
  - etc

Logarithmic time

- $f(n) = \lg_c n$
- logarithm definition:
  - $x = \log_c n$ if and only of $c^x = n$
  - by definition, $\log_c 1 = 0$

- In algorithm analysis we use the ceiling to round up to an integer
  - the ceiling of x (the smallest integer >= x)
  - e.g. $\text{ceil}(\log_b n)$ is the number of times you can divide n by b until we get a number <= 1
  - e.g.
  - $\text{ceil}(\log_2 8) = 3$
  - $\text{ceil}(\log_2 10) = 4$

- Notation: $\lg n = \log_2 n$

- Refresher: Logarithm rules

Exercises

Simplify these expressions

- $\lg 2n =$
- $\lg (n/2) =$
- $\lg n^3 =$
- $\lg 2^n$
- $\log_4 n =$
- $2^{\lg n}$

Binary search

- Searching a sorted array

```
//return the index where key is found in a, or -1 if not found
public static int binarySearch(int[] a, int key) {
    int left = 0;
    int right = a.length-1;
    while (left <= right)  {
        int mid = left + (right-left)/2;
        if (key < a[mid]) right = mid-1;
        else if (key > a[mid]) left = mid+1;
        else return mid;
    }
    //not found
    return -1;
}
```

- running time:
  - best case: constant
  - worst-case: lg n

  Why? input size halves at every iteration of the loop

Linear running time

- f(n) = n

- Example:
  - doing one pass through an array of n elements
  - e.g.
  - finding min/max/average in an array
  - computing sum in an array
  - search an un-ordered array (worst-case)

```
int sum = 0
for (int i=0; i< a.length; i++)
    sum += a[i]
```

n-lg-n running time

- $f(n) = n \lg n$

- grows faster than n (i.e. it is slower than n)
- grows slower than n^2

- Examples
  - performing n binary searches in an ordered array
  - sorting

Quadratic time

- $f(n) = n^2$
- appears in nested loops
- enumerating all pairs of n elements
- Example 1:

```
for (i=0; i<n; i++)
    for (j=0; j<n; j++)
        //do something
```

- Example2:

```
//selection sort:
for (i=0; i<n; i++)
    minIndex = index-of-smallest element in a[i..n-1]
    swap a[i] with a[minIndex]
```

- running time:
  - index-of-smallest element in a[i..j] takes j-i+1 operations
  - n + (n-1) + (n-2) + (n-3) + … + 3 + 2 + 1
  - this is n^2

Math refresher

- Lemma:
  - $1 + 2 + 3 + 4 + \ldots + (n-2) + (n-1) + n = n(n+1)/2$ (arithmetic sum)
- Proof:

Cubic running times

- Cubic running time: f(n) = n3
- In general, a polynomial running time is: f(n) = nd, d>0
- Examples:
  - nested loops
  - Enumerate all triples of elements
  - Imagine cities on a map. Are there 3 cities that no two are not joined by a road?
    - Solution: enumerate all subsets of 3 cities. There are n chose 3 different subsets, which is order n3.

Exponenial running time

- Exponential running time: $f(n) = an$, $a > 1$

- Examples:
  - running time of Tower of Hanoi (see later)
    - moving n disks from A to B requires at least 2n moves; which means it requires at least this much time

- Math refresher: exponent rules:

Comparing Growth-Rates

- $1 < \lg n < n < n \lg n < n^2 < n^3 < a^n$

Asymptotic analysis

- Focus on the growth of rate of the running time, as a function of n
- That is, ignore the constant factors and the lower-order terms
- Focus on the big-picture
- Example: we'll say that 2n, 3n, 5n, 100n, 3n+10, n + lgn, are all linear

- Why?
  - constants are not accurate anyways
  - operations are not equal
  - capture the dominant part of the running time

- Notations:
  - Big-Oh:
    - express upper-bounds
  - Big-Omega:
    - express lower-bounds
  - Big-Theta:
    - express tight bounds (upper and lower bounds)

Big-Oh

- Definition: f(n) is O(g(n)) if exists c >0 such that f(n) <= c g(n) for all n >= n0

- Intuition:
  - big-oh represents an upper bound
  - when we say f is O(g) this means that
    - f <= g asymptotically
    - g is an upper bound for f
    - f stays below g as n goes to infinity

- Examples:
  - 2n is O(n)
  - 100n is O(n)
  - 10n + 50 is O(n)
  - 3n + lg n is O(n)
  - lg n is O(log_10 n)
  - lg_10 n is O(lg n)
  - 5n^4 + 3n^3 + 2n^2 + 7n + 100 is O(n^4)

Big-Oh

- $2n^2 + n \lg n + n + 10$
  - is $O(n^2 + n \lg n)$
  - is $O(n^3)$
  - is $O(n^4)$
  - is $O(n^2)$

- $3n + 5$
  - is $O(n^{10})$
  - is $O(n^2)$
  - is $O(n+\lg n)$

- Let's say you are 2 minutes away from the top and you don't know that. You ask: How much further to the top?
  - Answer 1: at most 3 hours (True, but not that helpful)
  - Answer 2: just a few minutes.

- When finding an upper bound, find the best one possible.

Exercises

Write Big-Oh upper bounds for each of the following.

- $10n - 2$
- $5n^3 + 2n^2 + 10n + 100$
- $5n^2 + 3n\lg n + 2n + 5$
- $20n^3 + 10n \lg n + 5$
- $3 n \lg n + 2$
- $2^{(n+2)}$
- $2n + 100 \lg n$

Big-Omega

- Definition:
  - f(n) is Omega(g(n)) if exists c >0 such that f(n) >= c g(n) for all n >= n0
- Intuition:
  - big-omega represents a lower bound
  - when we say f is Omega(g) this means that
    - f >= g asymptotically
    - g is a lower bound for f
    - f stays above g as n goes to infinity
- Examples:
  - 3nlgn + 2n is Omega(nlgn)
  - 2n + 3 is Omega(n)
  - 4n^2 +3n + 5 is Omega(n)
  - 4n^2 +3n + 5 is Omega(n^2)

Big-Theta

- Definition:
  - f(n) is Theta(g(n)) if f(n) is O(g(n)) and f is Omega(g(n))
  - i.e. there are constants c′ and c″ such that c′ g(n) <= f(n) <= c″ g(n)

- Intuition:
  - f and g grow at the same rate , up to constant factors
  - Theta captures the order of growth

- Examples:
  - 3n + lg n + 10 is O(n) and Omega(n) ==> is Theta(n)
  - 2n^2 + n lg n + 5 is Theta(n^2)
  - 3lgn +2 is Theta(lgn)

Asymptotic Analysis

- Find tight bounds for the best-case and worst-case running times
- Running time is Omega(best-case running time)
- Running time is O(worst-case running time)

- Example:
  - binary search is Theta(1) in the best case
  - binary search is Theta(lg n) in the worst case
  - binary search is Omega(1) and O(lg n)
- Usually we are interested the worst-case running time
  - a Theta-bound for the worst-case running time
- Example:
  - worst-case binary search is Theta(lg n)
  - worst-case linear search is Theta(n)
  - worst-case insertion sort is Theta(n^2)
  - worst-case bubble-sort is O(n^2)
  - worst-case find-min in an array is Theta(n)
- It is correct to say worst-case binary search is O(lg n), but a Theta-bound is better

Asymptotic Analysis

- Suppose we have two algorithms for a problem:
  - Algorithm A has a running time of O(n)
  - Algorithm B has a running time of O(n^2)

- Which is better?

Asymptotic Analysis

- Suppose we have two algorithms for a problem:
  - Algorithm A has a running time of Theta(n)
  - Algorithm B has a running time of Theta(n^2)

- Which is better?
  - order classes of functions by their oder of growth
  - Theta(1) < Theta(lg n) < Theta(n) < Theta(nlgn) < Theta(n^2) < Theta(n^3) < Theta(2^n)
  - Theta(n) is better than Theta(n^2)
  - etc

  - Cannot distinguish between algorithms in the same class
    - two algorithms that are Theta(n) worst-case are equivalent theoretically
    - optimization of constants can be done at implementation-time

Order of growth matters

- Example:
  - Say n = 10^9 (1 billion elements)
  - 10 MHz computer ==> 1 instruction takes 10^-7 seconds
  - Binary search would take
    - Theta(lg n) = lg 10^9 x 10^-7 sec = 30 x10^-7 sec = 3 microsec
  - Sequential search would take
    - Theta(n)= 10^9 x 10^-7 sec = 100 seconds
  - Finding all pairs of elements would take
    - Theta(n^2) = (10^9)^2 x 10^-7 sec = 10^11 seconds = 3170 years
  - Imagine Theta(n^3)
  - Imagine Theta(2^n)

Order of growth matters

n	lg n	n	n lg n	n^2	n^3	2^n
8	3	8	24	64	512	256
16	4	16	64	256	4,096	65,536
32	5	32	160	1,024	32,768	4,294,967,296
64	6	64	384	4,096	262,144	1.8 x 10^19
128	7	128	896	16,384	2,097,152	3.40 x 10^38
256	8	256	2.048	65,536	16,777,216	1.15 x 10^77
512	9	512	4,608	262,144	134,217,728	1.34 x 10^154
1024	10	1024				
1024^2	20	1,048,576				
10^9						

- Assume we have a 1 GHz computer.
- This means an instruction takes 1 microsecond (10^-9 seconds).

- We have 3 algorithms:
- A: 400n
- B 2n^2
- C: 2^n

- What is the maximum input size that can be solved with each algorithm in:
  - 1 second
  - 1 minute
  - 1 hour

Running time (in microseconds)	1 sec	1 min	1 hour
400n			
2n^2			
2^n			

Exercise

- We have an array X containing a sequence of numbers. We want to compute another array A such that A[i] represents the average X[0] + X[1] + … X[i]/ (i+1).
  - `A[0] = X[0]`
  - `A[1] = (X[0] + X[1])/ 2`
  - `A[2] = (X[0] + X[1] + X[2]) / 3`
  - …
- The first i values of X are referred to as the i-prefix of X.

  X[0] + … X[i] is called prefix-sum, and A[i] prefix average.
- Application: In Economics. Imagine that X[i] represents the return of a mutual fund in year i. A[i] represents the average return over i years.
- Write a function that creates, computes and returns the prefix averages.

  ```
  double[] computePrefixAverage(double[] X)
  ```
- Analyze your algorithm (worst-case running time).

Asymptotic Analysis: Overview

- Running time = number of instructions
  - RAM model of computation
- Want the worst-case running time as a function of input size
  - the largest number of instructions on an input of size n
- Find the tight order of growth of the worst-case running time
  - a Theta-bound

- Classification of growth rates

  Theta(1) < Theta(lg n) < Theta(n) < Theta(nlgn) < Theta(n^2) < Theta(n^3) < Theta(2^n)

- At the algorithm design level, we want to find the most efficient algorithm in terms of growth rate
- We can optimize constants at the implementation step