

Algorithms
Computer Science 140 & Mathematics 168
Instructor: B. Thom
Fall 2004

Homework 9b

Due on Tuesday, 11/02/04 (beginning of class)

1. **[25 Points] Minimum Spanning Trees from the Shortest Paths Perspective!**

Show that a minimum spanning tree (MST) of an undirected graph is equivalent to the *bottleneck shortest paths* tree (BSPT) for the graph. A BSPT is a tree that spans the graph that has the following property: For every pair of vertices u and v , it contains a path connecting them whose longest edge is as short as possible.

2. **[25 points] Negative-Cost-Finding With Floyd-Warshall!**

In this problem, you will show how to modify Floyd-Warshall so that it can detect the presence of a negative-cost cycle (NCC).

- (a) Describe your modification. Rather than writing pseudocode, just describe in English what your cycle detection criterion is. This criterion should amount to a claim of some sort, e.g. some condition *foo* is **equivalent** to the graph having an NCC.
- (b) Demonstrate your algorithm's correctness by carefully proving this claim (as we did for Bellman-Ford in class, you must prove both directions).

3. **[25 points] An Algorithmic Get-Rich-Quick Scheme!**

Your job at My-I'm-Soft was fun for awhile, but your reputation has gotten out and you've been hired back by the brokerage firm of Weil, Proffet, and Howe at a whopping salary. (We won't be specific about the actual amount, which is to say that your very happy that the company writes its payroll checks on a 64-bit computer!).

Weil, Proffet, and Howe has just entered the **arbitrage** business. Arbitrage is a money-making scheme involving anomalies in international currency exchange rates. For example, imagine that 1 U.S. dollar buys 0.8 Zambian kwachas, 1 Zambian kwacha buys 10 Mongolian tughriks, and 1 Mongolian tughrik buys 0.15 U.S. dollars. Then, by converting currencies, a trader can start with 1 U.S. dollar and buy $0.8 \times 10 \times 0.15 = 1.2$ U.S. dollars. By capitalizing on such anomalies quickly (before they're detected and corrected by the markets), huge amounts of money can be made.

We'll assume that we're given n currencies and the exchange rate between every pair of currencies. That is, we are given currencies c_1, \dots, c_n and an $n \times n$ table R such that one unit of currency c_i buys $R[i, j]$ units of currency c_j . (The values for all the diagonal elements are one; if one were to exchange a currency with itself one gains or loses nothing. Further, negative values for $R[i, j]$ never occur.)

- (a) Lets begin with the following problem. Given a list of currencies and exchange rates and a particular currency c_i —i.e. a single source—determine the maximum

amount of each currency that you can obtain, beginning with 1 unit of currency c_i . For this part of the problem, assume that there exist no cycles that allow you to get arbitrarily rich via arbitrage.

- i. Describe how to modify Bellman-Ford so that it can perform this task. Your algorithm can be described in English (pseudo-code is not necessary).
 - ii. Describe in a sentence or two why your algorithm is correct. A detailed proof (like one seen for Bellman-Ford in class) is not needed. Rather by identifying what invariant is maintained, you should be able to appeal to our argument in class.
 - iii. Analyze your modified algorithm's runtime (again, borrowing results from class is fine).
- (b) Now assume that the exchange rates are such that it may be possible to get rich via arbitrage. That is, there may exist a cycle of currency exchanges that allows you to make more of your starting currency than you had initially.
- i. Describe how you could modify Bellman-Ford to determine if it is possible to profit from arbitrage with the given set currencies and exchange rates (again, pseudo code is fine).
 - ii. Carefully prove that your modified algorithm is correct. Don't rely on results from class here, rather reconstruct the proof shown in class so that it applies to this modified setting.
- (c) Hopefully this problem has convinced you that being able to print out some NCC in a graph when one exists is a pretty powerful thing to be able to do! However, we won't ask you to come up with such a procedure in this homework. You might wish to think about how one might do this, though. If there is enough interest, we can discuss ways to do this in class.