An edge quadtree for external memory

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Outline

1. The problem
2. Preliminaries
3. Our algorithm
4. Empirical evaluation
    - Application: Map overlay

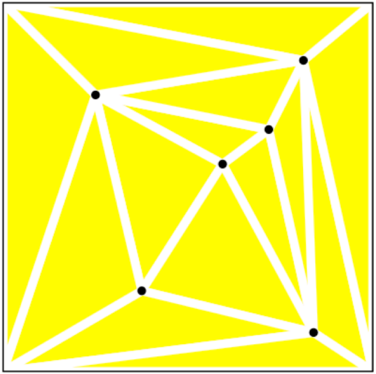
The problem

Build a quadtree subdivision for a set of non-intersecting edges in the plane.

Goal: Scalable to very large data (IO-efficient)

Quadtree: divide unit square into quadrants, refine until data per cell is “small”

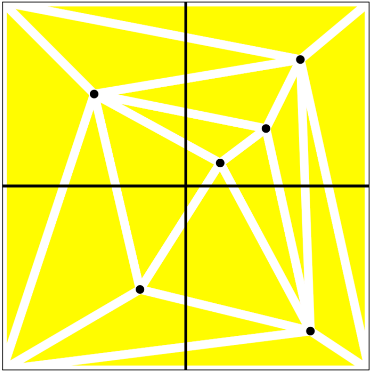
(for e.g., until every cell has at most one vertex)



figures thanks to H. Haverkort

Quadtree: divide unit square into quadrants, refine until data per cell is “scriptsize”

(for e.g., until every cell has at most one vertex)



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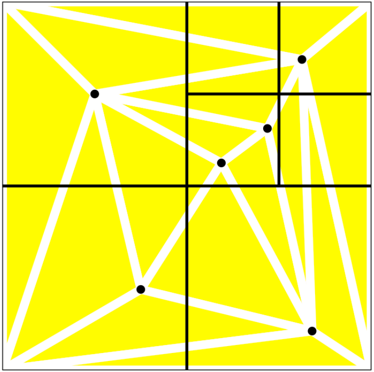

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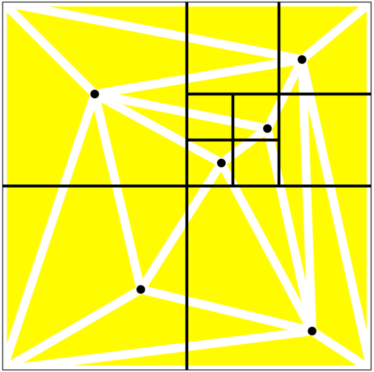

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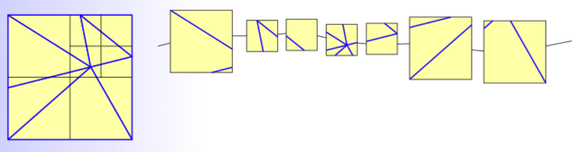
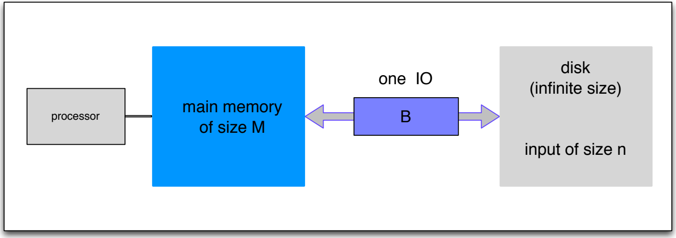
Linear edge quadtree

Linear (edge) quadtree: set of leaf cells, each cell storing its data.



- space efficient

The IO-Model [Agarwal & Vitter, 1988]



- IO-complexity: number of IOs
- Fundamental bounds
  - scan: $\texttt{scan}(n) = \frac{n}{B}$ IOs
  - sort: $\texttt{sort}(n) = \Theta(\frac{n}{B} \log_{M/B} \frac{n}{M})$ IOs

Quadtrees: Related work

Point quadtree

- compressed: $O(n)$ cells, $O(1)$ points each

Edge quadtree

- Build quadtree induced by endpoints and distribute edges
  - $O(n)$ cells, $O(1)$ points per cell, $I = O(n^2)$ edge-cell intersections
- Specific stopping criteria
  - E.g. split a region until it intersects a single edge (unbounded size)

Quadtrees: Related work

- Samet et al ('85,'86,'89,'92,'97,'99,'02)
  - PM quadtree (PM1, PM2, PM3)
  - segment quadtree
  - PMR quadtree
- Agarwal et al. 2006
  - build point qdt with $O(k)$ points per cell in $O(\frac{n}{B}\frac{h}{\log M/B}$ IOs
  - this is $O(\text{sort}(n))$ IOs when points are nicely distributed
- De Berg et al. 2010
  - star-qdt, guard-qdt
  - $O(1)$ point per cell, $O(\text{sort}(n+l))$ IOs
  - exploit fatness/low density: $O(1)$ edges per cell, $l = O(n)$

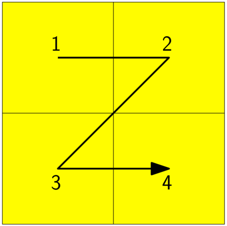
Our contributions

Let $\mathcal{E}$ be a set of n non-intersecting segments in the plane.
Let $k (k \geq 1)$ be a user-defined parameter.

- K-quadtree
  - Build a compressed, linear quadtree on the endpoints of E with $O(n/k)$ cells and $O(k)$ points in each cell in $O(\text{sort}(n))$ IOs.
  - Compute the intersections between the edges and the quadtree subdivision in $O(\text{sort}(n + I))$ IOs.
- Empirical evaluation
  - triangulated terrains, TIGER data

Quadtrees and Z-order

Z-order space filling curve: visits quadrants recursively in order NW, NE, SW, SE



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Quadtrees and Z-order

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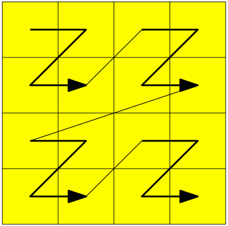

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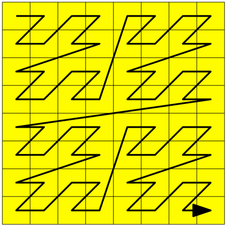

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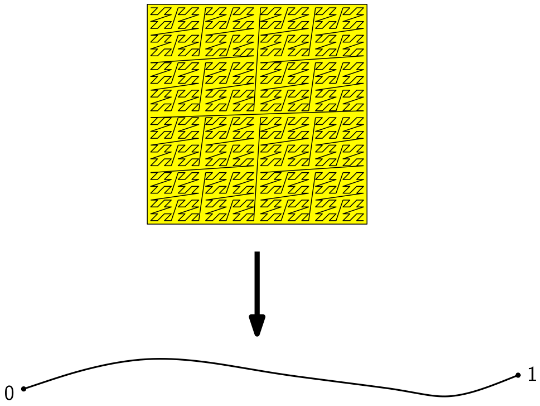

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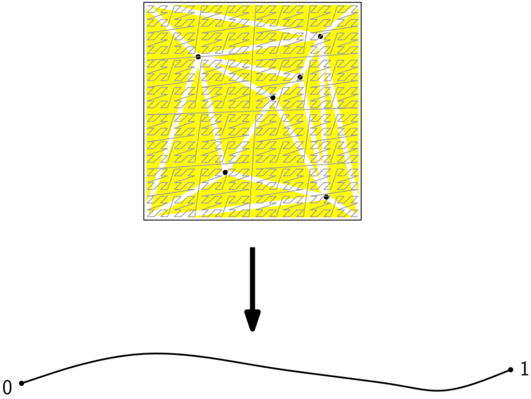
Quadtrees and Z-order

Quadtree cell = interval on Z-order curve

Quadtree subdivision = subdivision of Z-order curve



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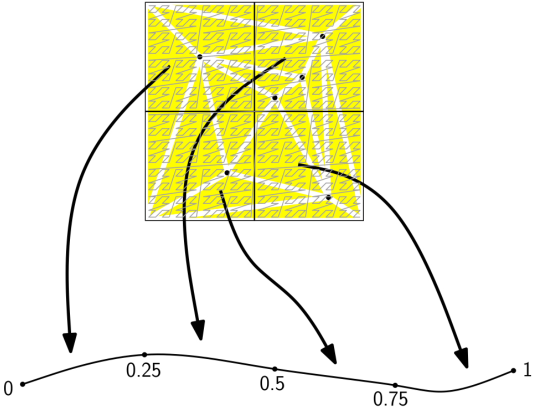

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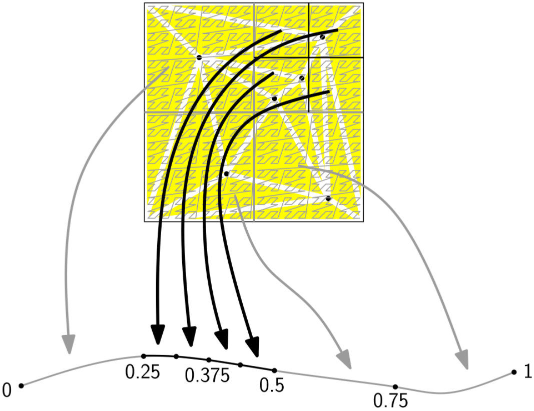

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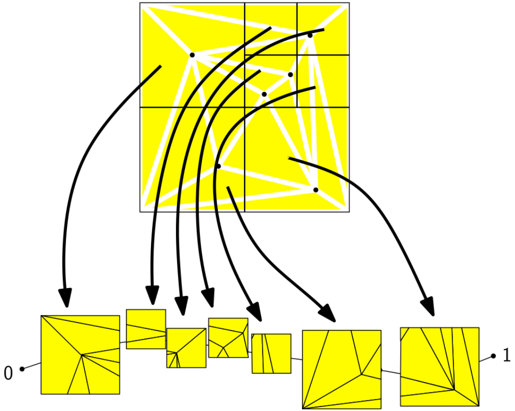
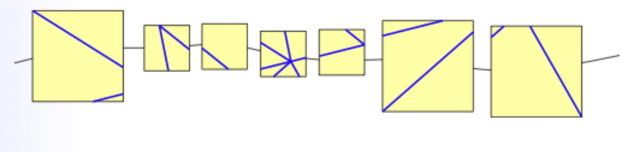


figures thanks to H. Haverkort

Our algorithm

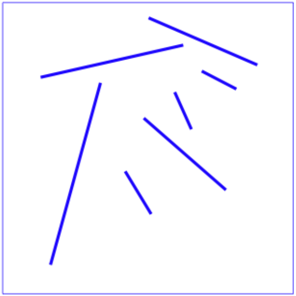
Input: Set of non-intersecting segments $\mathcal{E}$ and a value $k \geq 1$.

1. Construct the subdivision of the endpoints of $\mathcal{E}$
$\implies Q = \{[z_1 = 0, z_2], [z_2, z_3], ...\}$ a subdivision of $[0, 1]$
2. Compute the edge-cell intersections
$\implies Q = \{...(z_1, e), ...\}$



Our algorithm: Constructing the subdivision

Input: Set of non-intersecting segments $\mathcal{E}$ and value k.

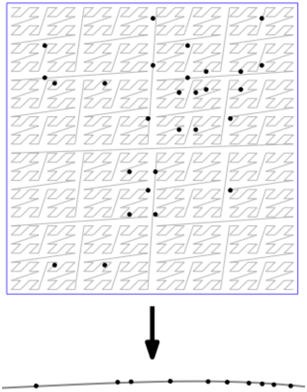


Our algorithm: Constructing the subdivision

Input: Set of non-intersecting segments $\mathcal{E}$ and value k.

Algorithm:

1. Find the endpoints of the segments and sort them in Z-order.

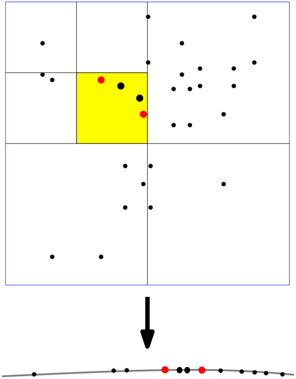


Our algorithm: Constructing the subdivision

Input: Set of non-intersecting segments $\mathcal{E}$ and value k.

Algorithm:

1. Find the endpoints of the segments and sort them in Z-order.
2. Let $P_k = \{p_0, p_k, p_{2k}, ...\}$ the set of every kth point. For every two consecutive points p and p' in P_k:
   - find smallest cell Q that contains p and p'



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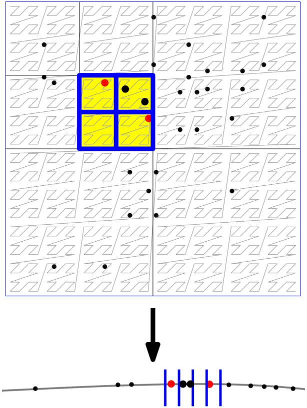
   - find smallest cell Q that contains p and p'
   - output cell boundaries of Q and its quadrants



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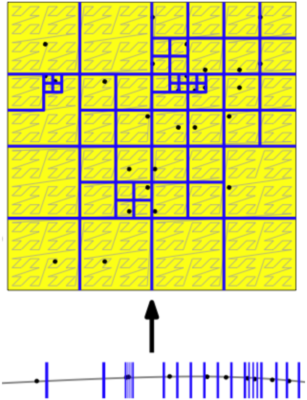
   - find smallest cell Q that contains p and p'
   - output cell boundaries of Q and its quadrants

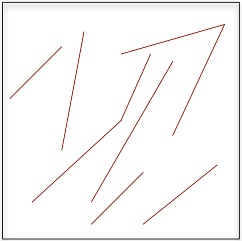
Lemma: Subdivision with $O(n/k)$ cells and $O(k)$ vertices per cell.



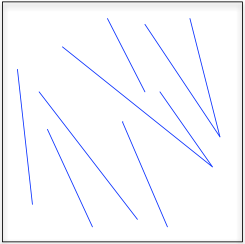
Our algorithm: Edge-cell intersection

Goal: For every $I_j = [z_j, z_{j+1}] \in Q$, compute the edges that intersect σ_j.

- Let E^+ (E^-) be the edges of positive (negative) slope



E+

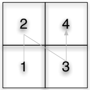


E-

Idea: Process E^+ and E^- separately.

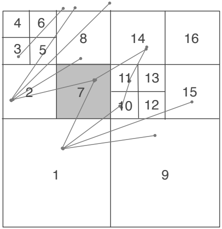
Finding the intersections of E^+ and Q ($k = 1$)

Input: $Q = \{[z_1, z_2], [z_2, z_3], ...\}$ in Z-order, E^+



Algorithm:

- For each interval $I_j = [z_j, z_{j+1}]$ in Q, find all edges in E^+ that intersect I_j



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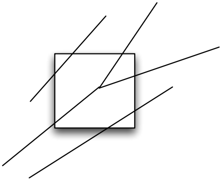


Algorithm:

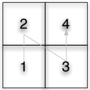
- For each interval $I_j = [z_j, z_{j+1}]$ in Q, find all edges in E^+ that intersect I_j

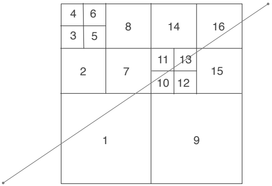
The edges that intersect I_j either:

- start in I_j or,
- start in an interval outside I_j

Finding the intersections of E^+ and Q ($k = 1$)

Input: $Q = \{[z_1, z_2], [z_2, z_3], ...\}$ in Z-order, E^+





Lemma

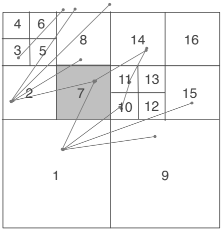
An edge of positive slope intersects the cells in Q in Z-order.

The edges that intersect I_j either:

- start in I_j, or,
- start in an interval before I_j

Finding the intersections of E^+ and Q ($k = 1$)

Input: $Q = \{[z_1, z_2], [z_2, z_3], ...\}$ in Z-order, E^+



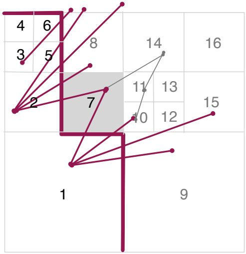
Algorithm:

- Sort E^+ by Z-index of first endpoint.
- For each interval $I_j = [z_j, z_{j+1}]$ in Q:
    - Find all edges in E^+ that originate in I_j: read all $e = (p, q)$ from E^+ s. th. $z(p) \in I_j$
    - Find all edges in E^+ that originate in an interval before I_j: HOW?

Finding the intersections of E^+ and Q ($k = 1$)

B_j: boundary between $\cup_{i<j}\sigma_i$ and $\cup_{i\geq j}\sigma_i$

BL_j: the edges that intersect B_j, in order.



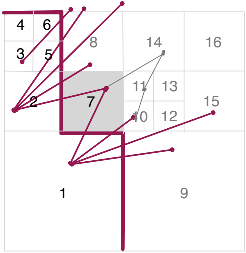
BL6

Finding the intersections of E^+ and Q ($k = 1$)

B_j: boundary between $\cup_{i<j}\sigma_i$ and $\cup_{i\geq j}\sigma_i$
BL_j: the edges that intersect B_j, in order.

Lemma

B_j is a monotone staircase and the intersection of σ_j and B_{j-1} covers a connected part of B_{j-1}.



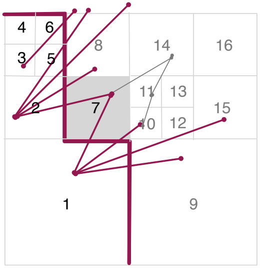
BL6

Finding the intersections of E^+ and Q ($k = 1$)

Input: $Q = \{[z_1, z_2], [z_2, z_3], ...\}$ in Z-order, E^+

Algorithm:

- Sort E^+ by z-index of first endpoint.
- For each interval $I_j = [z_j, z_{j+1}]$ in Q:
  - Find all edges in E^+ that start in I_j: read all $e = (p, q)$ from E^+ s. th. $z(p) \in I_j$
  - Use BL_{j-1} to find the edges that start before σ_j and intersect σ_j, and update BL_{j-1} to BL_j



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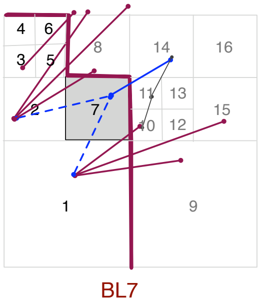

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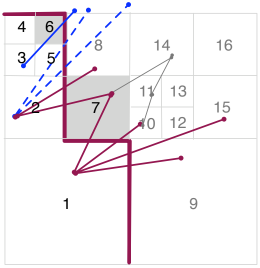
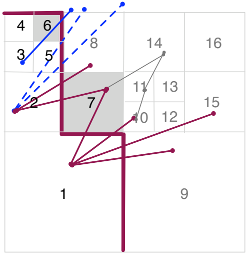


Search in BL_{j-1}: $\Omega(1)$ IOs per cell $\Longrightarrow \Omega(n)$ IOs
Idea: Start scanning BL_{j-1} from an edge that intersects σ_{j-1}

Finding the intersections of E^+ and Q ($k = 1$)



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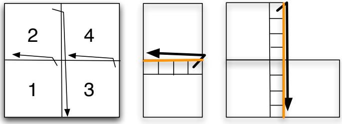


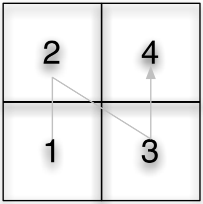
Lemma

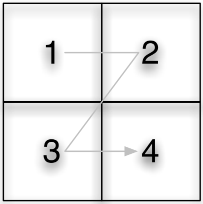
The number of edges traversed and skipped is $O(l)$, where l is the number of edge-cell intersections.

The intersections of E^+ and Q can be found in $O(\mathrm{scan}(n + l))$ IOs, once Q and E^+ are sorted.

Our algorithm

The algorithm generalizes to $k > 1$ and to E^-.





Empirical evaluation

`QDT-K`

- our algorithm for constructing a k-quadtree

Platform

- C
- Intel 2.83 GHz, 5400 rpm SATA drive (HP blade servers)
- 512 MB RAM

Datasets:

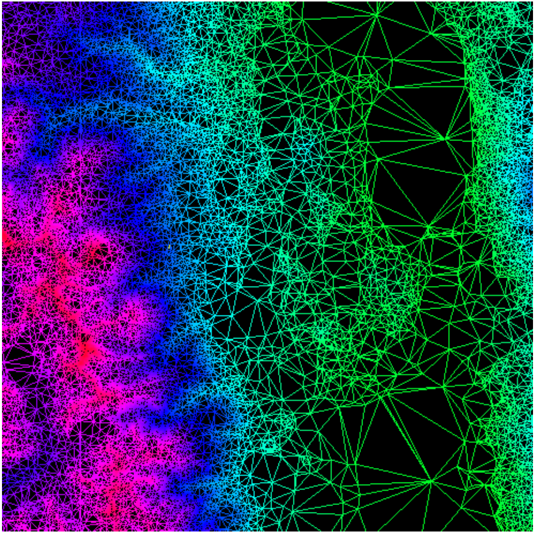
- triangulated terrains (TINs)
- TIGER data

Datasets: TINs

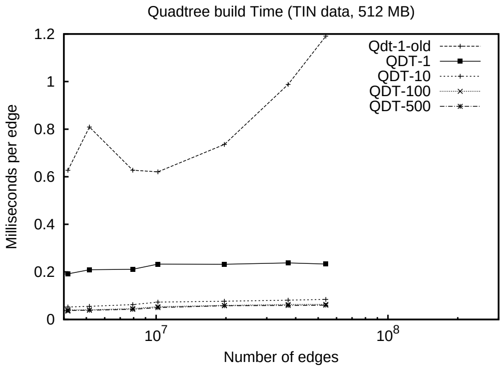
- We ignored the elevation.
- Delaunay triangulation
- Lots of small angles on the boundary

Dataset	e	Max inc.	Min $\angle$
Kaweah	$1.2 \cdot 10^6$	31	.0704
Puerto Rico	$4.1 \cdot 10^6$	291	.0010
Cumberlands	$5.1 \cdot 10^6$	44	.0016
Sierra	$7.9 \cdot 10^6$	75	.0137
Central App.	$10.1 \cdot 10^6$	62	.0013
Hawaii	$19.7 \cdot 10^6$	356	.0007
Haldem	$37.1 \cdot 10^6$	78	.0097
Lower NE	$53.9 \cdot 10^6$	168	.0021

Datasets: TINs



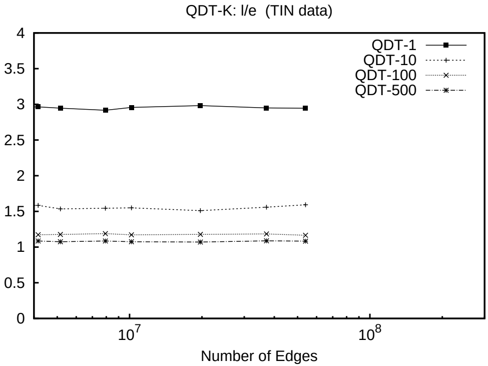
Results: TIN data



- `Qdt-1-old` from De Berg et al. 2010
- As *k* increases: construction time of `QDT-K` decreases

Results: TIN data

QDT−k total size



- As k increases: c decreases, and I/e decreases (fewer cells → fewer edge-cell intersections)

Results: TIN data

Sizes and build time on LowerNE ($e = 53.9 \cdot 10^6$)

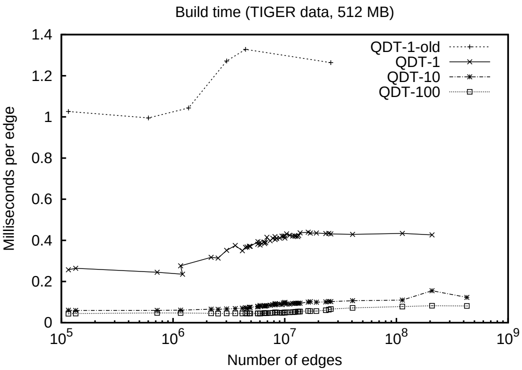
	c	l	l/c	build (min)
QDT-1-OLD	$32.5 \cdot 10^6$	$158.8 \cdot 10^6$	4.8	1,071
QDT-1	$32.5 \cdot 10^6$	$158.8 \cdot 10^6$	4.8	210
QDT-100	$.24 \cdot 10^6$	$62.8 \cdot 10^6$	257.4	57
QDT-500	$.06 \cdot 10^6$	$58.4 \cdot 10^6$	957.4	53
QDT-1000	$.02 \cdot 10^6$	$57.5 \cdot 10^6$	2456.5	54

Datasets: TIGER data

- Available at `http://www.census.gov/geo/www/tiger/`
- 50 sets, one set for each state, containing roads, hydrography, railways and boundaries
- Largest set: TX ($e = 40.4 \cdot 10^6$)
- We assembled larger bundles.

Dataset	e
`New England`	$25.8 \cdot 10^6$
`East Coast`	$113.0 \cdot 10^6$
`Eastern Half`	$208.3 \cdot 10^6$
`All USA`	$427.7 \cdot 10^6$

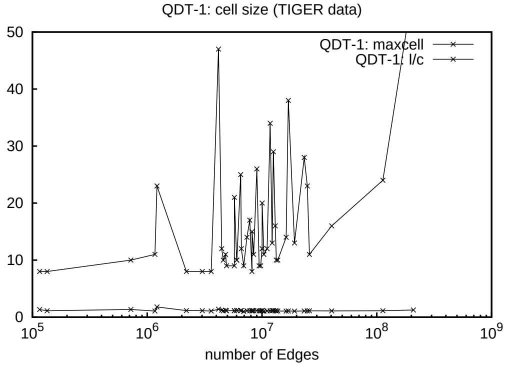
Results: TIGER data



- QDT−k gets faster up to $k = 100$ and then levels
- QDT−100 on AllUSA in 9.7 hours, 70% CPU.
- Bottleneck is finding edge-cell intersections

Results: TIGER data

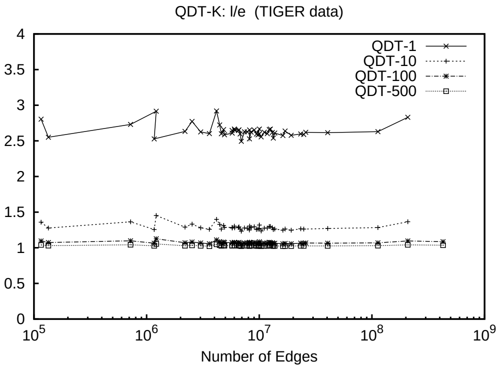
`QDT-1` cell size



- Varies widely from state to state
  - Easthalf: max cell intersects 58 edges
  - ME: max cell intersects 8 edges

Results: TIGER data

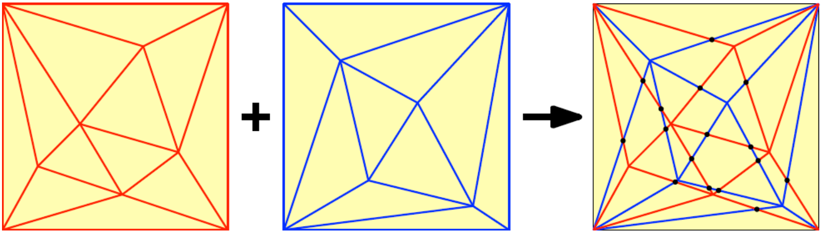
`QDT-k` total size



- As k increases: number of cells decreases, average size of a cell increases, and overall quadtree size decreases

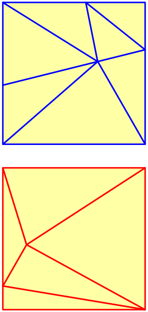
Application: Map overlay

Finding pairwise intersections between two sets of edges

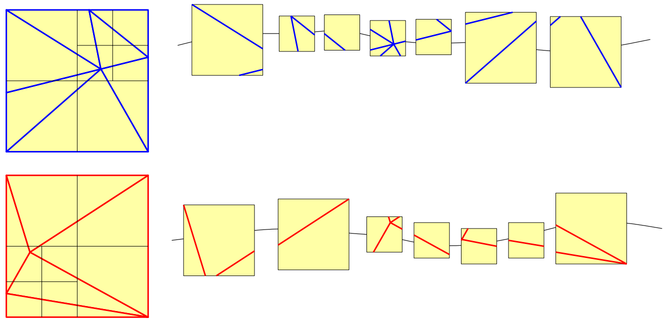


figures thanks to H. Haverkort

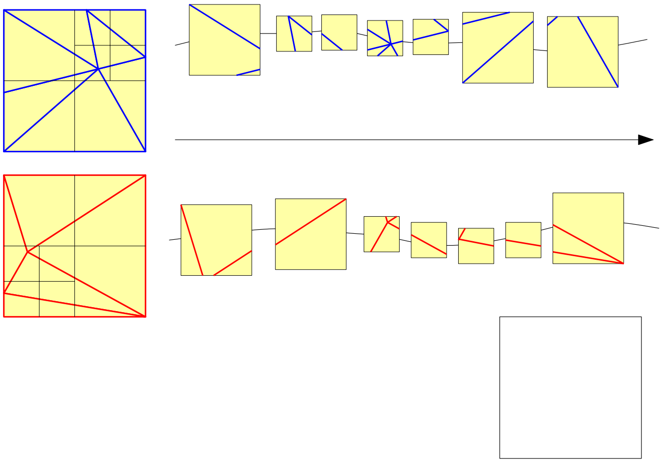
Map overlay with quadtrees in Z-order



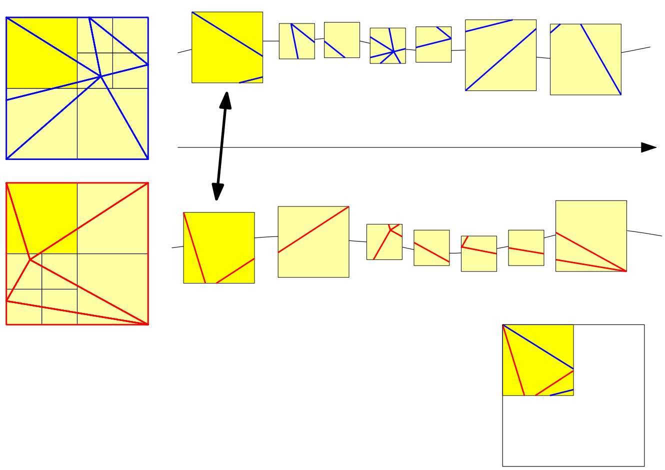
Map overlay with quadtrees in Z-order



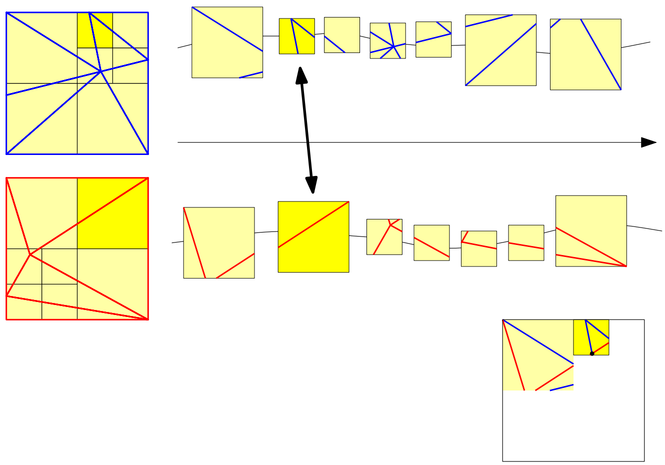
Map overlay with quadtrees in Z-order



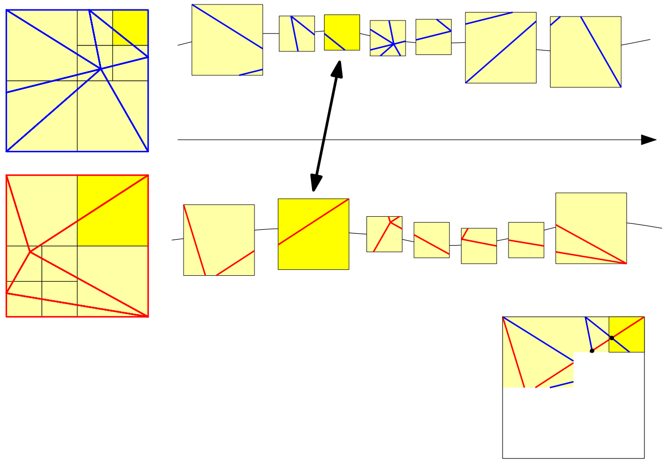
Map overlay with quadtrees in Z-order



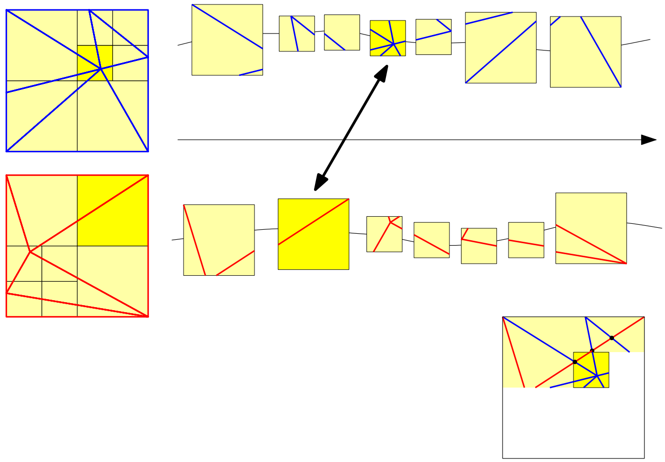
Map overlay with quadtrees in Z-order



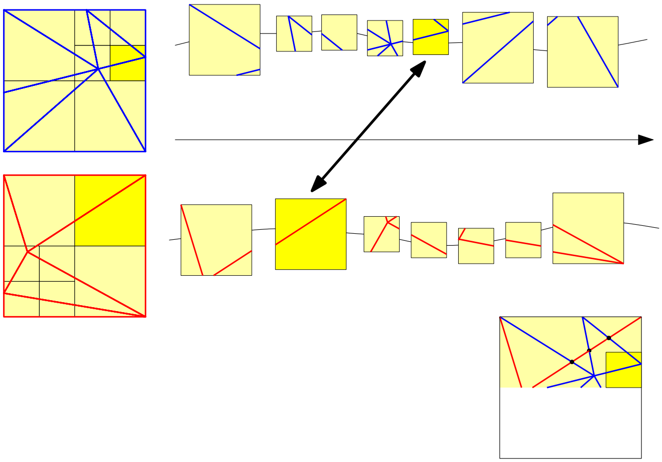
Map overlay with quadtrees in Z-order



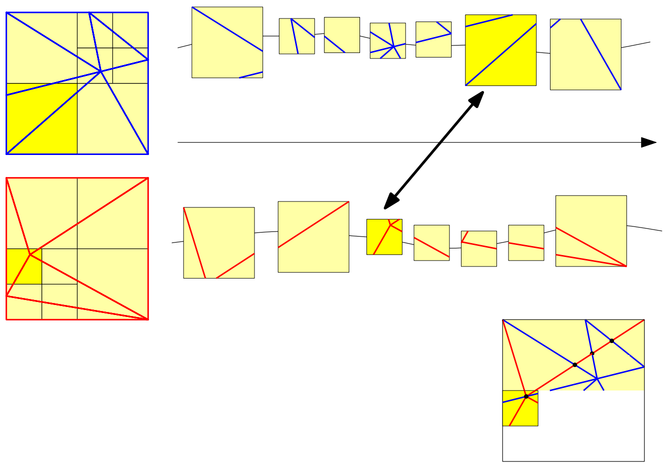
Map overlay with quadtrees in Z-order



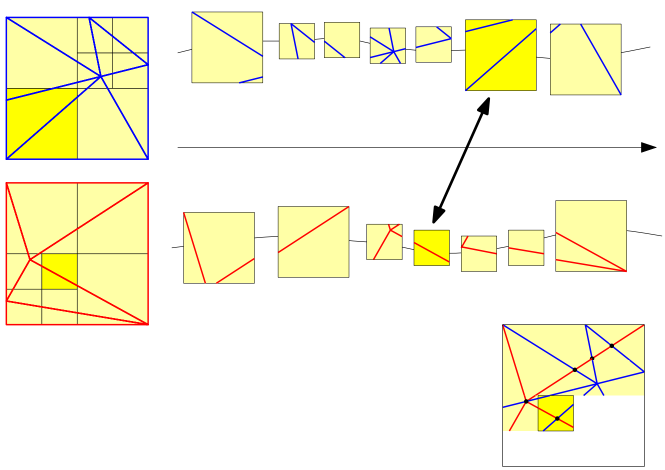
Map overlay with quadtrees in Z-order



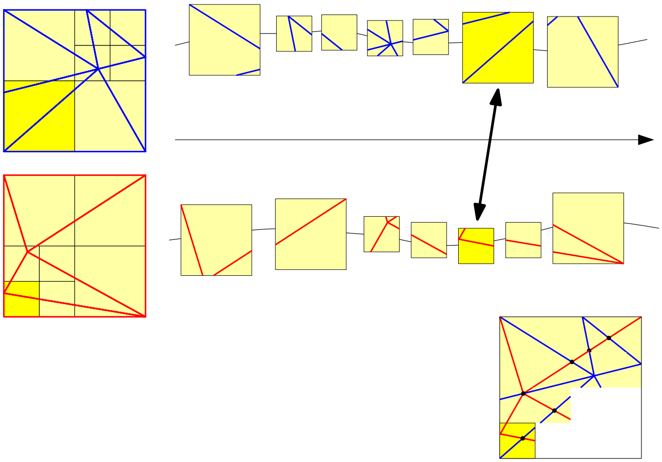
Map overlay with quadtrees in Z-order



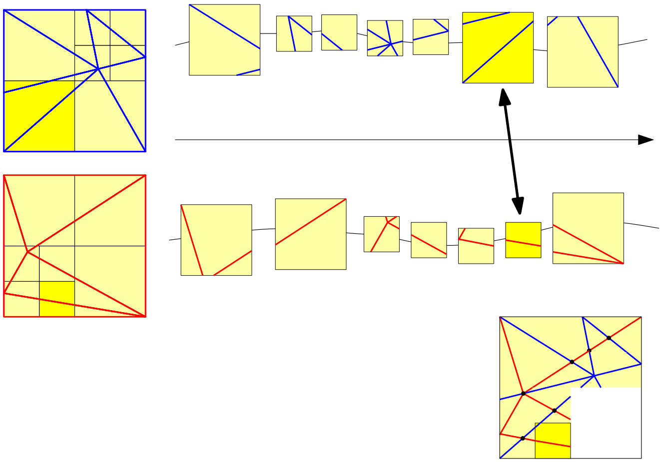
Map overlay with quadtrees in Z-order



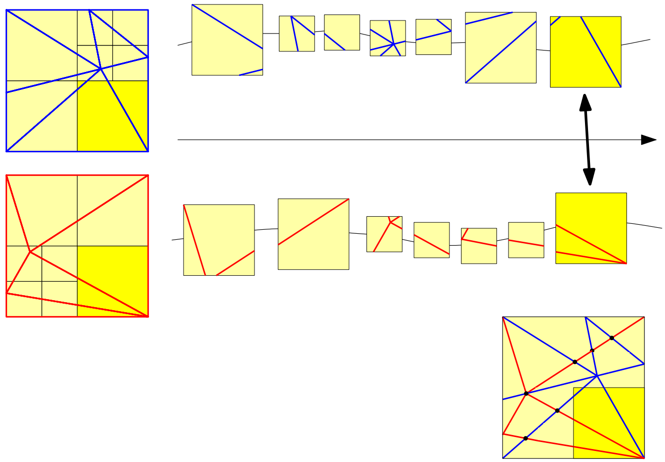
Map overlay with quadtrees in Z-order



Map overlay with quadtrees in Z-order



Map overlay with quadtrees in Z-order



Results: Map overlay with Quadtrees

- We overlayed a TIN stored as `QDT-1` with TIGER data stored as `QDT-k`
  - All data scaled to unit square
  - Artificial results
- Competing effects
  - size of a cell, and time for cell-cell intersection both increase with k
  - nb. of cells decreases with k
- Optimal k: $k \in [100, 500]$
- Overlay 262 mill. edge in 1.8 hours, if quadtrees are given

Conclusions

- New, IO-efficient algorithm for computing K-quadtrees with $O(k)$ vertices per cell, $O(n/k)$ cells, in $O(\text{sort}(n+I))$ IOs.
- Viable for two types of data used in practice
- As k increases, K-quadtrees are faster to compute and have smaller size.
- Optimal value k depends on application

Thank you!