The Francis $1-2-3$ Theorem. Start with a positive integer r and produce an N-digit number J such that, beginning with r in the "ones" position, the other digits in J increase in increments of k as they fill higher orders of "tens" positions in J (e.g. $r=3$, $N=5$, $k=1$, then $J=76,543$).

Find the sum of J and its *reverse*, $\hat{J}$ and denote that sum Q (e.g. $J=123$, then $\hat{J}=321$, and $Q=123+321$). Also find the sum of the digits in J (or $\hat{J}$) and denote that sum q.

Statement: *The quotient of these two sums Q/q does not vary under any choice for r or k, and depends only on the choice for N*

Proof. The integers J and $\hat{J}$ can be described in a more general form as series:

$$J = \sum_{n=0}^{N-1} [((N-1)k + r) - kn]10^n$$
$$\hat{J} = \sum_{n=0}^{N-1} (r + kn)10^n$$

and the sum of the digits in J (or $\hat{J}$) can also be put in series form:

$$q = \sum_{n=0}^{N-1} (r+kn) = \sum_{n=0}^{N-1} r + k\sum_{n=0}^{N-1} n = Nr + \frac{k(N-1)N}{2}$$

The sum $J+\hat{J}$ can be written

$$Q = J + \hat{J} = \sum_{n=0}^{N-1} [(N-1)k + r + (r+nk) - nk]10^n = \sum_{n=0}^{N-1} [(N-1)k + 2r]10^n$$
$$= [(N-1)k + 2r] \sum_{n=0}^{N-1} 10^n$$

so the quotient of the two sums

$$Q/q = \frac{[(N-1)k+2r]\sum_{n=0}^{N-1} 10^n}{Nr + [(N-1)k(N)/2]} = \frac{2[r+(N-1)k/2]\sum_{n=0}^{N-1} 10^n}{N(r+(N-1)k/2)} = (2/N)\sum_{n=0}^{N-1} 10^n$$

This shows that the quotient, depends only on N, which concludes the proof.