The Francis $1-2-3$ Theorem. Start with a positive integer r and produce an N-digit number J such that, beginning with r in the "ones" position, the other digits in J increase in increments of 1 as they fill higher orders of "tens" positions in J (e.g. $r=3$, $N=5$, then $J=76,543$).

Find the sum of J and its *reverse*, $\hat{J}$ (e.g. $J=123$, then $\hat{J}=321$). Now also find the sum of the digits in J (or $\hat{J}$).

Statement: *The quotient of these two sums does not vary under any choice for r, and depends only on the choice for N*

Proof. The integers J and $\hat{J}$ can be described in a more general form as series:

$$J = \sum_{n=0}^{N-1} [((N-1)+r)-n]10^n$$
$$\hat{J} = \sum_{n=0}^{N-1} (r+n)10^n$$

and the sum of the digits in J (or $\hat{J}$) can also be put in series form:

$$\sum_{n=0}^{N-1}(r+n) = \sum_{n=0}^{N-1} r + \sum_{n=0}^{N-1} n = Nr + \frac{(N-1)(N)}{2}$$

The sum $J+\hat{J}$ can be written

$$J+\hat{J} = \sum_{n=0}^{N-1}[(N-1)+r+(r+n)-n]10^n = \sum_{n=0}^{N-1}[(N-1)+2r]10^n$$
$$= [(N-1)+2r]\sum_{n=0}^{N-1} 10^n$$

so the quotient of the two sums

$$\frac{[(N-1)+2r]\sum_{n=0}^{N-1}10^n}{Nr+[(N-1)(N)/2]} = \frac{2[r+(N-1)/2]\sum_{n=0}^{N-1}10^n}{N(r+(N-1)/2)} = (2/N)\sum_{n=0}^{N-1}10^n$$

This shows that the quotient, depends only on N, which concludes the proof.