CSCI 189 Assignment 3 Problem Set

This assignment povides an opportunity to learn about some important alternative proof techniques beyond direct proof. Especially important in computer science applications are "proof by contradiction" and "proof by induction."

Summary of Concepts in Section 2.1-2.2

Direct Proof

The proof techniques we have studied so far are called "direct proof." That is, you begin with an argument of the form $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$, assume that its hypotheses $P_1, P_2, \ldots, P_n$are true, and try to find a direct path to the conclusion Q.

Another direct proof technique is "proof by exhaustion" - that is, you try out the argument under all possible values for the variables. This isn't usually a practical technique (see p 86 for a discussion).

Another variation of direct proof is a more informal one, in which some intermediate steps can be skipped because they are too trivial to write down (see p 87-88 for a discussion of this idea). Informal proofs are usually preferred in mathematics and computer science problem solving.

The following proof methods are more informal than those we have seen in Assignments 1 and 2.

Proof by Contraposition

To prove an argument $P \rightarrow Q$ by *contraposition*, you try to prove the argument $Q' \rightarrow P'$ instead. We an do this because these two expressions are logically equivalent.

For example, suppose we want to prove the argument "If 27 different passwords are issued to the 26 students in this class, then someone will have two passwords."

Let $P =$ "27 different paswords were issued to the 26 students in the class."

Let $Q =$ "someone has two passwords."

Then we can prove $P \rightarrow Q$ by assuming Q' and proving that this leads to the validity of P'.

Let's assume, therefore, that "nobody has two passwords." Then the total number of passwords issued can not be greater than 26 (the number of students in the class). In particular, that number cannot be 27, which is equivalent to P'.

Proof by Contradiction

This method is similar to proof by contraposition, but is not completely the same. That is, to prove the validity of the argument $P \to Q$, we assume the contrary $(P \to Q)'$ (or, equivalently $P \wedge Q'$) and then we show that this assumption leads to a contradiction. Recall that a contradiction is a proposition that is false in *all* circumstances.

To solve the above problem by contradiction, we would assume that "27 different passwords were issued to 26 students and nobody has 2 passwords." This, of course, is self-contradictory.

For another example, suppose we want to prove by contradiction that if a list of integers $[l_1, l_2, \ldots, l_n]$ is ordered, then $\forall i \in \{1, \ldots, n-1\} : l_i \leqslant l_i + 1$. Let's assume the contrary, that the list of integers $[l_1, l_2, \ldots, l_n]$ is ordered and $\exists i \in \{1, \ldots, n-1\} : l_i > l_i + 1$. This is self-contradictory, since there can be no such list with both these properties.

A third example is discussed in your text (p 90), which proves by contradiction that the product of two odd integers is an odd integer. Assuming the contrary, suppose that there were two odd integers x and y whose product xy is even. That is, suppose that:

$\exists m(x = 2m + 1)$ and $\exists n(y = 2n + 1)$, but that $\exists k(xy = 2k)$.
With these assumptions, we must find a value for k that satisfies $2k = (2m + 1)(2n + 1)$.
But $(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$, which is odd.

Thus our assumption leads to the contradiction that there is a number which is both even and odd.

Proof by Induction

This method of proof is very important in computer science, and we shall return to it often throughout the semester. To prove an argument of the form $\forall n P(n)$, the domain of n must be countable, like the integers, or lists of integers, and so forth. The strategy is to construct the proof in two steps:

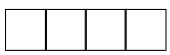
1. Prove P(1). This is called the *basis* step.
2. Assuming that $P(k)$ is true for an arbitrary k in the domain of n, prove $P(k + 1)$. That is, you have proved that $\forall k(P(k) \to P(k + 1))$. This is called the *induction* step. and the assumption $P(k)$ is called the *induction hypothesis.*

Here is an example. Suppose we want to prove by induction that the sum of the first n odd integers is n^2. That is, we want to show that $1 + 3 + \ldots + (2n - 1) = n^2$.

1. The *basis* step is easily shown, since $1 = 1^2$

2. For the *induction* step, assume as our *induction hypothesis* that $1 + 3 + \ldots + (2k - 1) = k^2$. Now we need to prove that $1+3+\ldots+(2(k+1)-1) = (k+1)^2$. But $1+3+\ldots+(2(k+1)-1) = 1+3+\ldots+(2k-1)+(2k+1) = k^2+2k+1 = (k+1)^2$, which completes the induction step.

Here is another example. Suppose you want to prove by induction that the number of distinct sides in a row of n adjacent squares is $3n + 1$. Here, for example, is a row of 4 adjacent squares, having 13 adjacent sides:



1. The *basis* step is easily shown, since 1 square has $3 \times 1 + 1 = 4$ sides (count 'em).

2. For the *induction* step, assume as our *induction hypothesis* that k squares have $3k + 1$ sides. Now we need to prove that this leads to the conclusion that $k + 1$ squares have $3(k + 1) + 1$ sides. But to construct a $k + 1$-square row, you simply add 3 sides to the k-square row. This leads to the conclusion that the number of sides in a $k+1$-square row is $3k+1+3 = 3(k+1)+1$, which completes the induction step.

Problems to be handed in

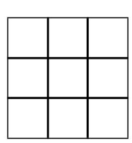
You are welcome to work in groups of 2 or 3 to complete this assignment. Each group member should contribute a fair share of the work, and the group should turn in one set of answers (listing the names of group members at the top).

Section 2.1 (p 92) Exercises 3ab, 6, 12, 17, 39, 44, 50.

Section 2.2 (p 105) Exercises 2, 8, 25bc, 30, 35, 50, 60, 64b, 68, 70, 71.

Extra credit (optional - some questions to ponder on a cold night when you have nothing better to do). Prove or disprove any of the following arguments:

1. Prove by induction that an $n \times n$ grid of adjacent squares has $2n(n + 1)$ sides. For example, a 3×3 grid has $2 \times 3 \times 4 = 24$ sides, as shown below:

2. Monkey language consists only of phrases like “abba dabba” and “abba dabba dabba dabba”. That is, all the phrases in the language are fully defined by the following two rules:
   (a) “abba” is a phrase in the language.
   (b) any phrasefollowed by “dabba” is also a phrase in the language.

Prove by induction that every phrase in monkey language is a palindrome. A “palindrome” is a phrase that is spelled the same way backwards and forwards, ignoring intermediate spaces. Here are some more examples of palindromes (ignoring the punctuation marks and capital letters):

```
May a moody baby doom a yam.
level
Hannah
racecar
civic
Rise to vote, sir!
Are we not drawn onward, we few, drawn onward to new era?
No lemons, no melon.
Able was I ere I saw Elba
detartrated
aibohphobia
Too bad, I hid a boot.
Was it a bar or a bat I saw?
Party boobytrap
Trapeze part
Yreka Bakery
A man, a plan, a canal Panama
```

Do you know any others?